Developing Students' Understandings of VARIABLE

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ALGEBRA'S "GATEKEEPER" STATUS HAS prompted several in the mathematics education research community (e.g., Davis 1985; Kaput 1998; Olive, Izsak, and Blanton 2002) to urge educators to view algebra not as an isolated course but as a continuous K–12 strand that is present throughout the entire mathematics curriculum. Central to the transition from arithmetic to algebraic reasoning is the concept of variable (Schoenfeld and Arcavi 1988). Schoenfeld and Arcavi argue that despite its importance, most mathematics curricula offer little to assist students in developing ideas about this concept. They assert that instead of providing students opportunities to practice manipulating terms and solving for unknowns, teachers should encourage students to view variables as shorthand tools for expressing already-understood ideas about varying quantities. This article describes a mathematical problem that can encourage students to view variables in this way while confronting a common misconception.

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Students' misconceptions about variables have been well documented. These misconceptions include viewing them as abbreviations or labels rather than as letters that stand for quantities (Booth 1988; Clement 1982; Weinberg et al. 2004), assigning values to letters based on their positions in the alphabet (MacGregor and Stacey 1997), and being unable to operate with algebraic letters as varying quantities rather than specific values. Booth further noted that many high school students believe that different letters within a number sentence must represent different numerical values.

Researchers with the Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR) project at the University of Wisconsin—Madison replicated Booth's (1988) findings by working with a sample of middle-grades students. As part of a larger assessment focusing on an array of issues related to student understanding of algebra (equality, variable, problem solving, representational fluency), 371 suburban middle school students were asked, "Is $h + m + n = h + p + n$ always, sometimes, or never true?" Results indicate that less than half of students in all grades correctly responded "sometimes" and that a significant number of students responded "never" (see **table 1**). A few elaborated by stating that "p is a different number than m" or "m and p cannot be the same number."

An accompanying interview was conducted with a subset of these students. Sixteen sixth-grade students were shown the number sentences $a = a$ and $c = r$ and were asked whether these number sentences were true or false and why. Most students (13 out of 16) stated that $a = a$ was true because, as

one student explained, "The variable *a* has to be the same in the same problem." Only 3 out of the 16 students, however, correctly stated that $c = r$ could be true or false, depending on the values of these variables. The majority of students held the belief that $c = r$ must be false, because, as one representative student responded, "When a letter represents a number, usually each letter represents a different number, not the same ones."

That these students demonstrated such a misconception should not be all that surprising given their relatively few prior experiences with variables. That the values of *c* and *r can* in fact be equivalent is a mathematical convention, not a notion that is intuitively obvious. Rather than simply being asked to memorize this information, however, students should engage in problem situations that support the adoption of this convention. The task discussed in this article presents one such situation.

The Mice Problem

AS PART OF A PROFESSIONAL DEVELOPMENT course that focused on the development of students' algebraic reasoning, STAAR researchers shared the aforementioned interview findings with fifteen sixth- through eighth-grade teachers (a few of whom taught the interviewed students). The demonstrated misconception was discussed, and then the following problem, adapted from Carpenter, Franke, and Levi (2003), was posed:

> Ricardo has 8 pet mice. He keeps them in two cages that are connected so that the mice can go back and forth between the cages. One of the cages is blue, and the other is green. Show all the ways that 8 mice can be in two cages.

(The original problem from Carpenter, Franke, and Levi 2003, p. 65, is as follows: "Ricardo has 7 pet mice. He keeps them in two cages that are connected so that the mice can go back and forth between the cages. One of the cages is big and the other is small. Show all the ways that 7 mice can be in two cages." The problem was changed from 7 mice to 8 mice to allow for the possibility of having the same number of mice in each cage.)

The teachers solved this problem on their own and shared their representations of the situation. Most teachers approached the problem by creating a table that systematically illustrated the nine possibilities (e.g., 0 mice in the green cage, 8 mice in the blue cage; 1 mouse in the green cage, 7 mice in the blue cage; and so on). The potential that this task offered for confronting students' misconceptions

TABLE 1
"Is $h + m + n = h + p + n$ Always, Sometimes, or Never True?" ($N = 371$)

	Grade 6	Grade 7	Grade 8
Always true	9.0%	9.6%	8.9%
Sometimes true	27.0%	36.8%	45.2%
Never true	26.2%	18.4%	27.4%
No response/don't know	34.4%	30.7%	16.3%
Other	3.3%	4.4%	2.2%

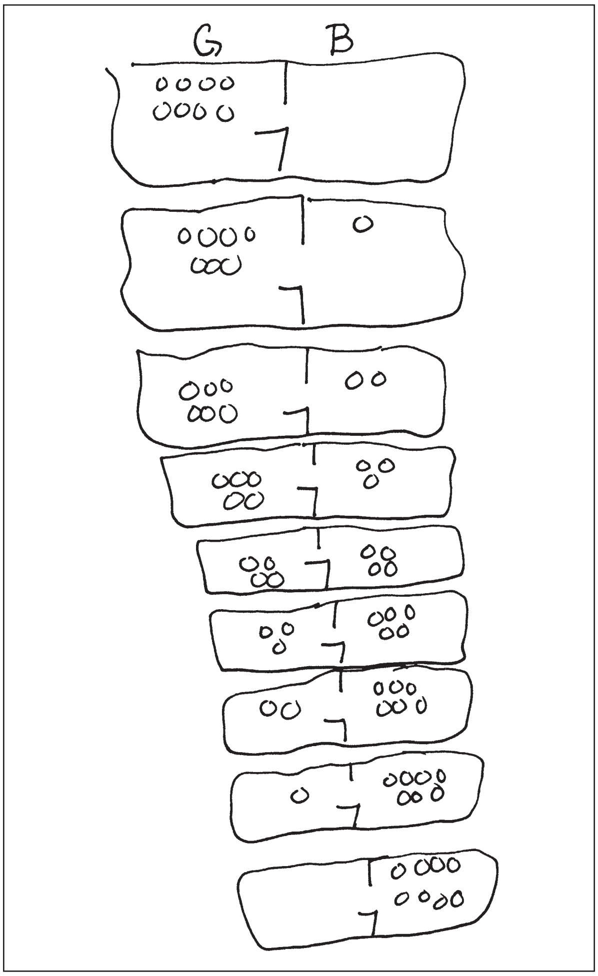

Fig. 1 A student's pictorial representation of the Mice problem

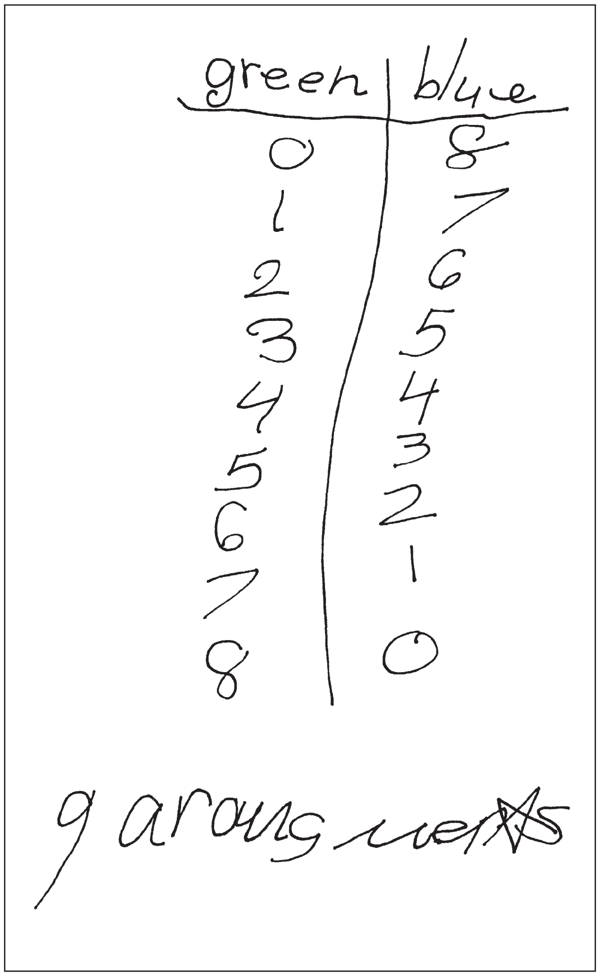

Fig. 2 A student's tabular representation of the Mice problem

about variables was then discussed. Teachers noted that the sum of the numbers of mice in the two cages was always equal to 8 and that the situation could be represented by the equation $b + g = 8$. One teacher then pointed out that in the case of 4 mice in each cage, the values of b and g would be the same and that this example could confront the aforementioned misconception.

The teachers were asked to pose the Mice problem to one of their mathematics classes and to bring the resulting student work to the next professional development meeting to share. Three of these teachers' classrooms were observed during the presentation of the task. After a brief summary of the group's overall findings, the discussion focused on these observations.

Student Representations of the Mice Problem

TEACHERS ACROSS THE GRADES FOUND THAT most students approached the Mice problem either by drawing pictures of the two cages and illustrating the different numbers of mice that could be in each (see **fig. 1**) or by creating tables listing the possible combinations of mice in the blue and green cages (see **fig. 2**). They found that many students were able to determine that there were nine possible ways to distribute 8 mice in two cages, but that several students did not take the possibility of having 0 mice in one cage and 8 in the other into account. Although the initial approach used by most students did not involve writing equations, the three observed teachers—Mrs. Rowley, Ms. Folberg, and Ms. Tuttle (all seventh-grade teachers)—made an effort to elicit multiple representations of the situation, including expressions involving variables.

After working on the Mice problem individually for some time, students in Mrs. Rowley's class presented unorganized lists of possible combinations, tables listing combinations in a more systematic manner, and drawings of the different ways that the 8 mice could occupy the two cages. One student then asked if she could present an equation she had written. She proposed that if b represented the number of mice in the blue cage and g represented the number of mice in the green cage, the equation $b = 8 - g$ could be used to represent the situation. This prompted another student to propose $b + g = 8$ and another to propose $g = 8 - b$.

Ms. Folberg's students likewise tended to illustrate the problem situation using tables and pictures. When prompted by Ms. Folberg to find more than one representation, some students created graphs illustrating how the number of mice in one cage increased while the number in the other decreased. A few of Ms. Folberg's students also generated an equation to describe the situation after being prompted to consider such a representation (see **fig. 3**).

In Ms. Tuttle's class, none of the students spontaneously generated an algebraic equation to represent the given situation. Most, however, were comfortable with the pictorial and tabular representations. Ms. Tuttle helped her students build from these representations to symbolic ones by asking, "What do you notice in this table? Do you see any patterns?" One student noted that the numbers down one column of the table decreased while those down the other column increased. Another student noted that the numbers in each row of the table summed to 8. When Ms. Tuttle asked the students if they could think of an equation that could be written to describe the situation, $8 \div 2 = 4$ was proposed. When asked if they could think of an equation that would apply to *all* possibilities, one student proposed $8 - g = b$. The class had been working on "fact families" and from this experience was able to generate the equivalent equations $8 - g = b$, $8 - b = g$, $b + g = 8$, and $g + b = 8$. The students agreed that these four equations were true for all the data generated for the Mice problem.

In the case of Ms. Folberg and Ms. Tuttle, the equations collectively generated by their students were explicitly used to confront the common misconception noted earlier. Ms. Folberg asked, for example, "Can $x = y$ [in $x + y = 8$] be true?" Some students initially said "no," some said "yes," and others said "sometimes." One student explained to the class that it was true when there were 4 mice in each cage.

Ms. Tuttle similarly asked her students what they thought about the statement $b = g$. They quickly agreed that it was sometimes true. When asked when the statement would be true, one student pointed to the situation of 4 mice in each cage. Ms. Tuttle stressed that this one time, two different letters—b and g—represented the same number. This mathematical convention appeared reasonable to the students when set in the context of the Mice problem.

Conclusion

MEMBERS OF THE STAAR PROFESSIONAL DEVELOPment course reported that the Mice problem provided rich opportunities for students to work with multiple mathematical representations. The student work that teachers collected and discussed in-

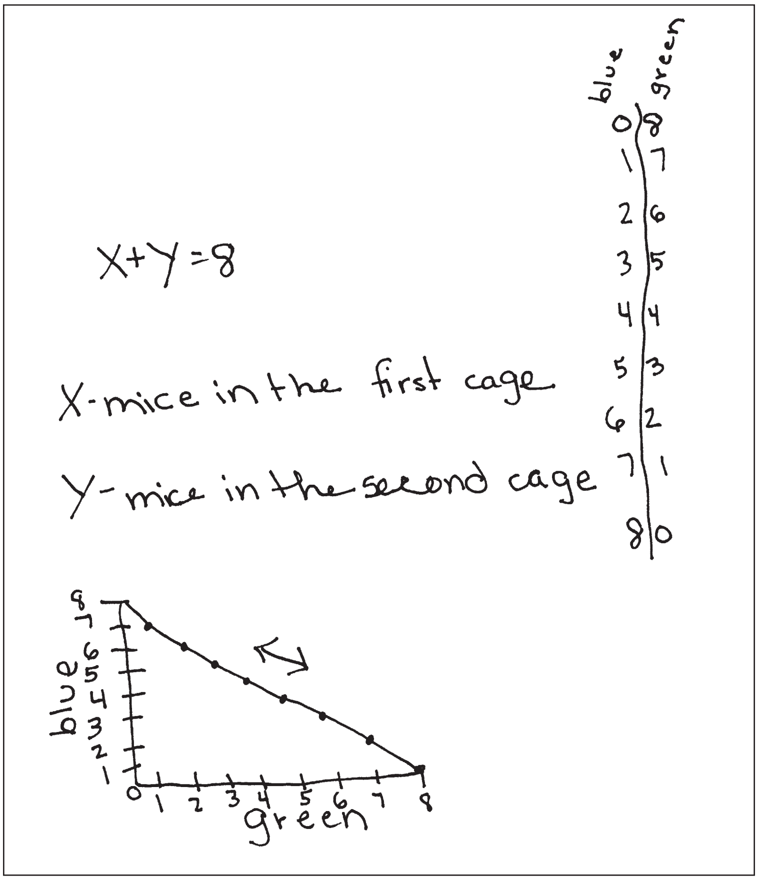

Fig. 3 A student's tabular, symbolic, and graphical representations of the Mice problem

cluded tables listing the possible combinations of mice in the blue and green cages, graphs showing the relationship between the number of mice in the two cages, and systematic drawings of the cages showing the number of mice in one cage increasing and the number in the other cage decreasing. Some students also represented the problem with algebraic equations, including $g + b = 8$, $b + g = 8$, $b = 8 - g$, and $g = 8 - b$.

Ms. Folberg and Ms. Tuttle additionally noted that focusing on such symbolic representations gave them the opportunity to discuss with their students that, in an equation such as $b + g = 8$, b and g can take on the same values. Such a discussion addresses a common misconception around the concept of variable.

In conclusion, the task discussed in this article accompanied by supporting classroom discussion can encourage students to employ and reflect on multiple mathematical representations. In the case of the symbolic representation, it can support student learning of an important mathematical convention—that it is possible for two different variables within an equation to take on the same numerical values. I encourage readers to try this task with their own students.

References

Booth, Lesley R. "Children's Difficulties in Beginning Algebra." In *The Ideas of Algebra, K–12*, edited by Arthur F. Coxford and Albert P. Schulte, pp. 20–32. Reston, VA: National Council of Teachers of Mathematics, 1988.

Carpenter, Thomas P., Megan L. Franke, and Linda Levi. *Thinking Mathematically: Integrating Arithmetic and Algebra in the Elementary School*. Portsmouth, NH: Heinemann, 2003.

Clement, John. "Algebra Word Problem Solutions: Thought Processes Underlying a Common Misconception." *Journal for Research in Mathematics Education* 13 (January 1982): 16–30.

Davis, Robert B. "ICME 5 Report: Algebraic Thinking in the Early Grades." *Journal of Mathematical Behavior* 4 (1985): 195–208.

Kaput, James J. "Transforming Algebra from an Engine of Inequity to an Engine of Mathematical Power by 'Algebrafying' the K–12 Curriculum." In *The Nature and Role of Algebra in the K–14 Curriculum: Proceedings of a National Symposium*, edited by Skip Fennell, pp. 25–26. Washington, DC: National Research Council, National Academy Press, 1998.

MacGregor, Mollie, and Kaye Stacey. "Students' Understanding of Algebraic Notation: 11–15." *Educational Studies in Mathematics* 33 (January 1997): 1–19.

Olive, John, Andrew Izsak, and Maria Blanton. "Investigating and Enhancing the Development of Algebraic Reasoning in the Early Grades (K–8): The Early Algebra Working Group." In *Proceedings of the Twenty-Fourth Annual Meeting of the International Group for the Psychology of Mathematics Education*, edited by Denise S. Mewborn, Paola Sztajn, Dorothy Y. White, Heide G. Wiegel, Robyn L. Bryant, and Kevin Nooney, pp. 119–20. Columbus, OH: ERIC, 2002.

Schoenfeld, Alan H., and Abraham Arcavi. "On the Meaning of Variable." *Mathematics Teacher* 81 (September 1988): 420–27.

Weinberg, Aaron D., Ana C. Stephens, Nicole M. McNeil, Daniel E. Krill, Eric J. Knuth, and Martha W. Alibali. "Students' Initial and Developing Conceptions of Variable." Paper presented at the Annual Meeting of the American Educational Research Association, San Diego, California, April 2004.

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