Students' Initial and Developing Conceptions of Variable

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A primary goal of the Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR) project is to characterize students' emerging algebraic thinking. Algebraic variables are an integral part of both the conventional algebra curriculum as well as less traditional curricula. For example, the text for the Connected Mathematics Project (CMP) (REF)—the curriculum used by students in our study—has roughly 250 problems and examples involving variables in the sixth-grade text and roughly 1500 problems and examples in the seventh-grade text. Both this ubiquity and their conceptual complexity suggest that algebraic variables should be considered in studying the development of students' algebraic reasoning. STAAR researchers have been analyzing students' evolving responses to written assessment and interview items while concurrently developing a framework to describe the ways in which students interpret algebraic notation and think about algebraic variables. The framework will also allow us to identify and classify students' misconceptions about representations of variables. Before reporting students' responses to a subset of our written assessment and interview items, we outline a collection of prior results and theories that inform our work.

The Framework For Analyzing Variable Use

We incorporate ideas from several studies and conceptual frameworks that describe students' interactions with variables. Along with several studies that deal with students' misconceptions about algebraic variables (see e.g. Kinzel, 1999), we integrate theories that can be classified into three general categories: psychological models, experiential models, and structural models. The framework is both "top-down" as well as "bottom-up;" while we use prior theory as a starting point to create items for our written assessments and interviews, we look at students' responses to these items to inform our understanding of prior theories, decide which elements to keep in the framework, and generate new aspects of the framework. As such, the framework is constantly evolving.

Psychological Component

For many years, the preeminent psychological model was one proposed by Kuchemann (1978). He classified student responses to questions involving variables into general categories and then ordered them into Piagetian levels. The resulting hierarchical categories described students' treatment of literal symbols while implicitly incorporating a structural hierarchy. For example, the category "letter used as a specific unknown" —in which the letter is interpreted as an unknown number with a fixed value—is described as a lower level than the category "letter used as a generalized number"—in which the letter is interpreted as taking on multiple values in turn. Kuchemann's descriptions of the uses of algebraic variables are useful because they provide a basis for constructing questions that elicit a wide range of student responses. We have, in fact, used his descriptions to generate our initial set of written assessment variable items. We have not, however, found his assertion of the existence of a strict Piagetian hierarchy—the basis of his analysis—to be compelling because it does not take into account the role of experience.

In contrast, Trigueros and Ursini (2003) explicitly divide letter use into three structural categories —"unknown number", "generalized number" and "functional relationship")—and then present levels of psychological conceptualization within each category. They justify the hierarchical structure by citing the demands of prerequisite knowledge. For example, Trigueros and Ursini claim that "pattern recognition" is the lowest level of their generalized number hierarchy because "A prerequisite in order to develop an understanding of variable as a general number, is the ability to recognize patterns and to find or deduce general rules and methods describing them" (p. 4). As with Kuchemann's study, we find Trigueros and Ursini's descriptions of their levels to be useful but feel that their psychological ordering does not take into account the important role of students' previous experiences of working with representations of variables.

The process-object frameworks developed by Dubinsky (1996), Sfard (1991) and Tall (1994) can be used to characterize some types of student interactions with variables and their representations. Dubinsky's Action-Process-Object-Schema framework, while widely used as a tool to analyze many aspects of student cognition, has previously been applied to student conceptions of variables in only a handful of studies (e.g. Trigueros, 1996). This framework can explain why a student may have trouble (for example) adding three to "a number divided by five" because "a number divided by five" (or $x/5$) must be conceptualized as an object that may be acted upon in order to add a number to it. This process-object idea can be very useful for describing student responses to similar items.

Experiential Component

The only major study (of which we are aware) that has explicitly described the role of students' experience and prior knowledge when interpreting literal symbols was conducted by MacGreggor and Stacey (1997). While generally accepting Kuchemann's (1978) framework, they provide numerous examples to demonstrate that students' experiences contribute to the ways in which they interact with variables. For example, MacGreggor and Stacey propose that experience with concatenation of symbols (concatenation indicates addition in arithmetic thinking) may prompt students to express "10 more than *h*." for example, as 10*h* in the same way that "five more than 20" would be expressed as "25." Although it would be impossible take into account all possible types of experiences involving emerging thinking about variables when creating a framework, we are attempting to incorporate a general sense of the experiences students have had to enable us to more clearly understand their interactions with variables.

The idea of "prototypes" or "concept image" (e.g. Vinner, 1983) can be employed as another tool to describe students' interactions with variables. As Trigueros and Ursini (2003) note of students in their study, "Superficial characteristics of the expressions seem to determine their decisions independently of the use of variable involved in the problem. Many of their actions appear to be provoked by external signs (for example, exponents, the equal sign, the way in which a question is posed) that lead them to respond in stereotyped forms." (p. 19). Although we are not aware of studies that have employed the idea of prototypes in the context of algebraic variables, they can certainly be effectively descriptive tools.

In addition to looking at other studies, the STAAR project is analyzing the ways in which variables are used in the CMP textbooks. We are analyzing the semiotic structure of the variables (how they are used and represented) as well as the kinds of tasks in which the students are asked to use them. We hope that this analysis will offer further insight into the role of experience in students' conceptions of variables.

Structural Component

It has already been noted that both Kuchemann (1978) and Trigueros and Ursini (2003) have used hierarchical models to describe increasingly sophisticated ways of interacting with variables. Usiskin (1988) has also proposed a structure to describe variable use. His model consists of four ways of using variables: as pattern generalizers used to translate and generalize patterns, as unknowns or constants used to simplify and solve equations, as arguments or parameters in functional relationships, or as abstract elements of algebraic structures such as groups and rings. Several researchers (e.g. Radford 2001, Presmeg 2001, Ernest 2002) have proposed that semiotic analysis can be a useful tool for describing mathematical symbol use and production. Although very few studies have actually attempted to do this for variables, this call seems to complement both the idea that students' prior experiences with representations of variables are important and the idea that we may describe variables structurally by the way they are used. We have used the structural models (proposed by Kuchemann, Trigueros and Ursini, and Usiskin) as a basis for our semiotic description of variables in our analysis of the CMP text. We incorporate these semiotic ideas into our framework by considering the ways in which we both use and represent algebraic variables (and the ways we interpret and create those representations).

Although a detailed description of our evolving framework is beyond the scope of this paper, we have attempted to unify these three components into a coherent analytical tool. We employ the framework in the discussion of our initial results in the sections to follow.

Participants and Instruments

Our written assessment was administered to 373 students (123 sixth-graders, 115 seventh-graders and 135 eighth-graders) in a middle school. Thirty-one sixth-graders participated a videotaped, semi-structured interview (all but four of the interview participants also completed the written assessment). These written assessment and interview items will be administered to the current sixth-graders for two more years to enable us to gather longitudinal data on their thinking about variable.

Using ideas from our evolving framework, we created general descriptions of student responses and then categorized these responses. All codes were checked for reliability with independent coders reaching at least 85% agreement. Although the interviews have yet to be formally coded, we provide some informal analysis of them in this paper to contribute to our analysis of the written items.

Items involving algebraic variables formed a proper subset of both the written assessment and interview items. We discuss three of the written assessment items and use results from two of the interview items to enrich our analysis. The first written assessment item was designed to elicit students' basic interpretations of what a letter could represent in a mathematical context:

> The following questions are about this expression:
>
> $$\underset{\uparrow}{2n} + 3$$
>
> a) The arrow above points to a symbol. What does the symbol stand for?
> b) Could the symbol stand for the number 4? <u>Please explain your answer.</u>
> c) Could the symbol stand for the number 37? <u>Please explain your answer.</u>
> d) Could the symbol stand for the expression 3r + 2? <u>Please explain your answer.</u>

Part a is designed to determine if students are familiar with the symbols, while part b is designed to determine if students believe *n* can represent a number. We ask if *n* can stand for 37 to determine if students think a single letter can represent a two-digit number. We ask if *n* can stand for $3r+2$ to determine if students think *n* can represent an expression that can be viewed as a process and if they view the *r* (in $3r+2$) as a process or an object.

Although the 6^{th} grade CMP text has problems and examples involving variables that are represented by letters, the written assessment was administered early in the school year and students may not have had much experience with variables represented by letters. In arithmetic reasoning, concatenation indicates place-value addition (so if *n* stands for 4, $2n$ would stand for 24). Since students may view *n* as representing a single place value, they may not think it could represent 37, which uses two place values. It is unlikely that students, especially in 6^{th} and 7^{th} grades (where they have not yet had much experience substituting expressions for variables), will believe *n* can represent $3r+2$. Process-object theories suggest that students initially view a variable as the act of substituting an "object" for the variable. If $3r+2$ indicates the process of addition or if *r* represents an act of substitution, it is not an "object" that can be substituted in place of *n*.

The second written assessment item was designed to determine whether students thought of letters as representing quantities or as abbreviations for words (such as *c* representing "cakes"), as they are commonly used outside of explicitly mathematical contexts:

> Cakes cost c dollars each and brownies cost b dollars each. Suppose I buy 4 cakes and 3 brownies. What does 4c + 3b stand for?

Students' prior experience could prevent them from producing a correct answer. We frequently use letters to represent words and this experience would lead many people to view $4c$ as representing "four cakes." Since the correct answer is given by the expression $4c+3b$ in its entirety (that is, one must somehow "join" the $4c$ and $3b$ which represent distinct parts of the total cost), students need either to view $4c+3b$ as an encapsulated quantity or to have experience with similar questions.

The third written assessment item was designed to determine if students could more easily conceptualize a varying quantity in a verbal form than in the standard notation with a literal symbol. Students were given one of the following two versions, with 123 students responding to the first ("symbolic") version and 250 responding to the second ("verbal") version.

> *Symbolic Version:*
> Can you tell which is larger, 3n or n + 6? Please explain your answer.
>
> *Verbal Version:*
> A friend gives you some money. Can you tell which is larger,
> the amount of money your friend give you plus six more dollars, OR
> three times the amount of money your friend gives you?
> Please explain your answer.

The idea of verbal advantage (Koedinger, Alibali & Nathan, 1999) predicts that students will be more successful with the verbal version, which is grounded in a context that should be familiar to most students.

The first interview item was nearly identical to the first written assessment item, but enabled the interviewer to probe for sources of misconceptions. Although several probes were specified, each interviewer was free to include additional probes as they deemed appropriate:

I'm going to ask you some questions about this expression.
*Clip card with 2*n *+ 3 to the paper.*

a) Have you seen mathematical expressions like this before?
If yes: Where have you seen them?

b) Draw arrow pointing to n from below.
Could you tell me what this symbol stands for?

c) Could the symbol stand for the number 37? *Write 37 on the paper.*
(Probe: Why/why not?)

d) Could the symbol stand for 27 + 15? *Write 27 + 15 on the paper.*
(Probe: Why/why not?)

e) Could the symbol stand for the expression 4r + 1? *Write 4r + 1 on paper.*
(Probe: Why/why not?)

The second interview question was designed to elicit students' "natural" ways of representing varying quantities.

I'm going to describe a situation and I'd like you to tell me how you could write it mathematically.
"I have some number of pencils and then get three more."
How would you write that mathematically?

a) Could you explain your notation to me? Why did you decide to write it this way?

b) What if, after getting the three more pencils, I get two more pencils? How many pencils do I have now?
Why did you decide to write it that way?

If student uses a letter or symbol such as a box to represent the number of pencils you started with, ask:
What does that [letter/symbol] stand for? Could you have used a different [letter/symbol]?

If student is comfortable with these representations, continue.

c) What if I start with some number of pencils and then get three pens? How could you write that mathematically?

(Probes: If student writes (exactly) the same expression [symbol] +3 for both parts a and c, point to their work in part a and ask:
I notice these are the same expressions, but in part a the expression was about pencils and now the expression is about pencils and pens. How can they be the same?)

(Probes: If student writes: [symbol] +3 for both parts a and c (but the symbol used in part e is different than the symbol used in part a), then ask:
What you just wrote looks similar to what you wrote in part a. Could you have written the expression [from part c] this way [point to the expression in part a]? Why or why not?)

d) *Point back to student's notation in part a while talking.* What if, after starting with some number of pencils and then getting three more, I doubled the number of pencils I have? How could you write that mathematically?

Results and Discussion

Problem 1

We found parts b, c, and d of the first written assessment item to be progressively more difficult for students. Overall, students' performance increased with grade, and older students experienced a smaller drop in performance from part b to part c. Part d, however, was difficult for all students. Table 1 shows the percentage of students who responded "Yes" by grade. Table 2 shows the percentage of students who indicated that n could stand for any number or expression (and in part d students who indicated that since r stands for a number, $3r+2$ is a number and n can stand for it) by grade. Finally, Table 3 shows the percentage of students who responded, "Yes" and justified their answer by indicating that n can represent any number or expression, again by grade.

Table 1 – Students Responding "Yes," by Grade

Grade	Can n stand for 4?	Can n stand for 37?	Can n stand for $3r+2$?
6	56%	30%	26%
7	77%	67%	30%
8	87%	81%	47%

Table 2 – Students Indicating *n* Can Represent Any Number or Expression, by Grade

Grade	Can n stand for 4?	Can n stand for 37?	Can n stand for $3r+2$?
6	35%	22%	11%
7	60%	55%	11%
8	73%	73%	22%

It is not surprising that students were progressively more likely to believe that n could represent any of these three possibilities as they got older. As students are introduced to variables (informally in sixth-grade and formally in seventh-grade) they become more comfortable using letters to represent quantities and transition from reasoning arithmetically to reasoning algebraically. From a semiotic perspective, the students experience new signs as they formally encounter variable and unknown signifieds and become experienced at interpreting letters as having meaning similar to "missing value" or "fill-in-the-blank" problems. From a psychological perspective, 3r+2 presents an additional hurdle to jump, as students must either rotely accept that a letter may stand for "any expression" or encapsulate these processes that involve adding two quantities into a single quantity or object that can be substituted for the letter. The expression $3r+2$ additionally involves multiplication of two quantities *within* the addition of two quantities, necessitating another act of encapsulating "3 times r" into a quantity. Furthermore, if a student views a variable as "the act of replacing the letter with a number," they must encapsulate the letter as the result of this substitution before they can encapsulate the entire expression $3r+2$ as a mathematical quantity. The significance of encapsulation is underscored by the following interview excerpt:

Interviewer: Have you seen mathematical expressions like this before? Where have you seen them?

Student: I think so. I think we did something like this in fifth grade.

I: What sort of things were you doing in fifth grade?

S: I just remember n because it was one of the variables my teacher used.

I: Okay. Could you tell me what this symbol stands for?

S: (No response)

I: You don't know? What if you had to guess, what would it stand for?

S: Um

I: How about this? Could n stand for the number 4?

S: It could because letters are also used as variables and so n could be any number.

I: Could n stand for the number 37? (Writes 37 on paper)

S: Yeah.

I: That would be okay? Great. Could n stand for 15 + 27?

S: I think so.

I: Why do you think so?

S: Because well actually I don't think so really because variables just stand for one number. You could have n plus another letter or variable and n could be 15 and the other variable could be 27.

I: Can you write down what you just said?

S: I think that n can stand for 15, so n = 15 (writes n=15) and p can equal 27 (writes p=27) so then if you did n + p would equal 42 (writes n + p = 42).

I: Okay. So could n stand for—I'm going to write this a little differently. What if I put 15 + 27 in parentheses? Could n stand for that?

S: Yes because when you have parentheses that means before you do anything else, you do whatever in the parentheses is first and what is in the parentheses stands for another number and so 15 plus 27 in the parentheses would stand for 42.

I: And that would be okay?

S: Yeah.

I: Could n stand for 4r + 1 in parentheses (writes (4r + 1))?

S: I don't think so because it couldn't really stand for well actually yes it could because it's all in parentheses so the r could stand for another number and then it would all be one number actually but inside the parentheses and so you'd have to do that and come up with an answer.

I: Okay what about (writes (4r) + 1)? Could n stand for that?

S: No.

I: How come?

S: Because what is in the parentheses stands for a separate number so and it's basically what you did up here (points to 15 + 27). It's as if you were doing like 10 plus 5 in parentheses and then plus 27 out of the parentheses.

I: Could n stand for the letter r all by itself?

S: I don't think it could because n, variables stand for numbers and if n were a variable then it would have to stand for a number, not another letter.

I: Would it make a difference if I put parentheses around it (writes (r) on paper)?

S: I think it might.

I: How come?

S: Because then n is standing for what r equals, not just r but I'm not quite sure.

I: So it seems okay if n stands for what r equals?

S: Yeah.

I: So that's sort of what you were saying up here (points to (4r + 1). You look at what this equals and it's okay for n to stand for that?

S: Yeah.

I: Okay, good thinking.

This student sees a distinction between an expression that represents a process—such as 27+15—and the result of that process. The distinction between *r* and (*r*)—"what *r* equals"—indicates that this student conceives of *r* as representing an action or process of replacing the symbol *r* with a value and not the encapsulated result of that process.

Table 3 - Students Justifying "Yes" by Indicating n Can Represent Any Number, by Grade

Grade	Can *n* stand for 4?	Can *n* stand for 37?	Can *n* stand for 3*r*+2?
6	60%	68%	35%
7	78%	81%	37%
8	85%	90%	45%

Students' inability to justify their "Yes" responses could be a result of their limited experience. They may not yet have a deep enough understanding of the situation to be able to verbalize their reason for answering "Yes." Alternatively, they may have developed a concept image of mathematical expressions and literal symbols that affords an answer of "Yes" but limits their understanding to only superficial aspects of the image.

Problem 2

As with problem 1, students' performance on problem 2 increased with grade, with older students were more likely to say that $4c+3b$ represented the total cost of cakes and brownies. Almost 60% of eighth-graders gave either this response or indicated that $4c+3b$ represented 4 times the cost of the cakes plus three times the cost of the brownies, while the percentage of sixth- and seventh-graders giving one of these responses were nearly 30% and 40%, respectively.

Table 4 shows the percentage of students in each grade who gave either one of these responses or indicated that $4c+3b$ represented “4 cakes and 3 brownies,” treating c and b as standing in for words. Students not represented in this table either gave no response, indicated that they didn’t know how to respond, or gave responses that could not be classified into statistically significant groups.

Table 4 - Student Responses by Grade

Grade	4 Cakes and 3 Brownies	4 Times the Cost of Cakes Plus 3 Times the Cost of Brownies	Total Cost
6	22%	17%	11%
7	37%	21%	18%
8	27%	33%	26%

Although Kuchemann claimed that students would need to have progressed to a particular developmental level in order to move beyond a "4 cakes and 3 brownies" response, we can also interpret these responses on semiotic, experiential and psychological axes. Focusing our attention on 4*c*, we may hypothesize about how a student might reason about this sign. In the statement of the question, the student is told that "*c*" is the "cost of cakes." The sign "*c*" has a clearly outlined signifier until it is concatenated with the 4. Now the student must draw on prior experience with concatenation and substitution to interpret 4*c* as both two separate signs with distinct meanings as well as a single sign that indicates the act of multiplication of two quantities The student must then encapsulate the result of this action into yet another sign that represents the result of that multiplication (although the symbol remains constant, the signifier changes). If the student does not recognize that this process involves a shift in the signified, they may be stuck in their interpretation of the multiplication-signified or attempt to erroneously re-interpret the multiplication. If the student can view 4*c* as a single sign but still interprets the sign as multiplication, or if they cannot encapsulate 4*c,* they may view the 4 and the *c* as still-distinct signs and interpret 4*c* as "four times what *c* represents." Alternatively, if the student can not encapsulate 4*c* as a quantity, they may rely on previous experiences with symbols like 4*c* and arrive at a reasonable interpretation of "four of some object that is represented by *c*."
If the student can view 4*c* as an encapsulated quantity, they must do the same thing for the sign 3*b* and then finally for the sign 4*c*+3*b*. Students may also have experience with mathematical situations and expressions like those in question 2 and may be able to use these images to construct "total cost" as a reasonable answer to the question. This triaxial analysis offers insight into the cognitive resources a student uses to interpret, engage in, and provide an answer for this problem, with the role of experience (with similar problems, similar signs and the process of encapsulation) explaining why older students provided more sophisticated answers.

Problem 3

Problem 3 produced interesting results when we compared student responses by grade and by form (with the "verbal" form stating the situation in words and the "symbolic" form giving a symbolic expression to analyze). As with the other written items, student responses increased in sophistication with their grade. While there was little difference across forms for seventh- and eighth-graders, the sixth-graders were significantly more successful at giving correct and sophisticated responses for the verbal form than for the symbolic form. Table 6 shows the percentage of students (by form and grade) who indicated that you couldn't tell if 3*n* or *n*+6 is larger. Table 7 shows the percentage of students (by form and grade) who viewed *n* as taking on multiple values (for example, stating that *n* could be both 1 and 10). Table 8 shows the percentage of students (again, by form and grade) who indicated that you couldn't tell which is larger and justified this by saying that *n* could take on multiple values. Nearly all students who indicated that one cannot tell which was larger viewed *n* as a variable except for sixth-graders who were given the symbolic form.

Table 6 - Students Responding "Can't Tell" by Grade and Form

Grade	Verbal	Symbolic
6	43%	18%
7	57%	54%
8	68%	64%

Table 7 - Students Viewing *n* as a Variable, by Grade and Form

Grade	Verbal	Symbolic
6	40%	10%
7	49%	51%
8	63%	60%

Table 8 - Students Justifying "Can't Tell" by Viewing *n* as a Variable, by Grade and Form

Grade	Verbal	Symbolic
6	89%	57%
7	86%	95%
8	95%	93%

In contrast to the first two written assessment items, a sophisticated response to this problem requires that the student view the varying quantity as just that—something that can not only take on multiple values, but must have those static values compared to each other. More explicitly, the amount of money or value of *n* can not only be 1 or 10—making the potential amounts of money \$3 and \$7 (when the amount of money is 0) or \$30 and \$16 (when the amount of money is 10)—but these value pairs must be simultaneously compared in order to realize that one cannot determine the answer from the given information.

The most striking aspect of students' responses is the performance of sixth-graders on the symbolic form of the question. While the seventh- and eighth-graders have experience working with symbolic variables, many of the sixth-graders have not yet formally used letters to represent either unknown numbers or varying quantities. However, these students are still able to conceive of varying quantities when the situation is sufficiently grounded within a context—a verbal advantage. They can access the mechanisms they have developed to deal with the situation and not only begin to analyze the possible amounts of money but also to realize that the outcome of situations like this probably depend on the details of the situation.

Insights From the Interviews

The first interview question prompted a wide range of responses. Sixth-graders indicated that they had encountered variables in many different capacities, from seeing them in older siblings' textbooks to their participation in extracurricular math clubs. They exhibited a wide range of responses to the four parts of this question, frequently demonstrating misconceptions such as the idea that a letter must represent only a single-digit number. Most interestingly, students responded differently when expressions were written inside parentheses. For example, a student who thought that *n* couldn't stand for 27+15 because it couldn't represent addition thought that *n could* stand for (27+15). We hypothesize that the inclusion of parentheses could help students encapsulate expressions and plan to systematically include the addition of parentheses in future interviews.

Students were generally successful with the second interview question, with all but two able to find a way to mathematically represent the situation "I have some number of pencils and then get three more." We found a positive correlation between responding "Can't tell" to written question 3 and using a letter to represent the "some number" of pencils, as well as a weak positive correlation between responding viewing the letter/unknown amount of money in written question 3 as a varying quantity and using a letter to represent the number of pencils.

Limitations

The most serious limitation on the written items was, of course, not knowing if the students' responses were a good indicator of their thoughts or understanding. This is underscored by the fact that students who provided unsophisticated answers on the written questions (such as writing "I don't know") were able to provide thoughtful answers to the same questions when interviewed. Similarly, student responses were limited by their attention span and patience during the interview.

Because we are generating our framework from student responses to the written and interview questions, our interpretations are constantly evolving. Although we are occasionally able to modify our items to reflect new theories, the longitudinal design of the experiment limits the extent of our modifications.

Since experience plays such an important role in shaping students' conceptions of algebraic variables and their representations, students who do not use CMP may conceive of variables differently. Finally, the students at the middle school where this study was conducted perform well on standardized tests, which may limit the generalizability to students at other schools.

Conclusions

Although our instruments and framework for interpreting and analyzing students' interactions with representations of algebraic variables are constantly evolving, we have been able to use what we have thus far constructed to provide insightful descriptions of students' responses. The psychological-experiential-structural framework allows us to harness powerful ideas from previous studies to provide insight into student performance on our written and interview items. We plan to continue integrating ideas of encapsulation and the role of experience into our analysis and interview prompts to make further connections between the longitudinal interview and written responses.

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