STUDENT UNDERSTANDING OF GENERALITY

Eric J. Knuth
University of Wisconsin-Madison
knuth@education.wisc.edu

Jamie Sutherland
University of Wisconsin-Madison
sutherla@math.wisc.edu

This paper presents results from a multi-year research project[1] exploring the development of middle school students' competencies in justifying and proving. In particular, we present and discuss results from a written assessment completed by 394 sixth through eighth grade students. The assessment questions that are the focus of this paper targeted the idea of generality—both the idea that a general argument (i.e., proof) offers an absolute guarantee regarding the truth of a statement or result and the idea that empirical evidence does not suffice as proof.

The nature and role of proof in school mathematics has been receiving increased attention in the mathematics education community with many advocating that proof should be a central part of the mathematics education of students at all grade levels (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002; Schoenfeld, 1994; Sowder & Harel, 1998). Such attention is also reflected in current mathematics education reform initiatives. In contrast to the status of proof in previous national standards documents, its position has been significantly elevated in the most recent document (National Council of Teachers of Mathematics [NCTM], 2000). In particular, the *Principles and Standards for School Mathematics* (NCTM) recommends that the mathematics education of pre-kindergarten through grade 12 students enable all students "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56). These recommendations, however, pose serious challenges for school mathematics students given that many students have found the study of proof difficult. In fact, research has painted a relatively bleak picture of students' understandings of proof (e.g., Balacheff, 1988; Bell, 1976; Healy & Hoyles, 2000; Porteous, 1990; Senk, 1985, Sowder & Harel).

Although students' difficulties with proving have been attributed to a variety of factors, one factor, an understanding of generality, is critical to developing an understanding of the concept of proof. One aspect of generality concerns the idea that a proof offers an absolute guarantee regarding the truth of a statement or result. A number of researchers have investigated student understanding with respect to this particular aspect; such studies have found that many students do not seem to have an understanding of this aspect of generality. For example, Chazan (1993), studying high school geometry students, found that some students viewed deductive proofs as verifications of single cases that were subject to possible counterexamples. Porteous (1990) examined the type of evidence students found to be convincing. His results indicated that when presented with a particular case, over half the students empirically checked it rather than appealing to the proof of the general case that they had previously been shown and had presumably accepted. Similarly, Fischbein and Kedem (1982) found that most of the students in their study opted for supplementary checks of an already proven statement, one with which they had previously expressed their full agreement. A second (related) aspect of generality concerns the idea that empirical evidence does not suffice as proof. Again, for many students this aspect of generality appears to be one that they do not adequately understand—a finding that predominates the results of many studies is students' reliance on the use of examples to prove the truth of a statement or result (e.g., Balacheff, 1988; Healy & Hoyles, 2000). For example, Healy and Hoyles, in their study of high attaining 14- and 15-year old students, found that empirically-based arguments dominated the nature of justifications students provided in response to the researchers' assessment questions. Chazan also reported a similar finding: students believed that empirical evidence allows one to make general claims about the truth of a proposition.

Although the aforementioned studies suggest that students do not have a very robust understanding of (either aspect of) generality, little research has specifically explored the nature of students' understandings of generality itself. The purpose of this paper is to provide insight regarding students' understandings of (the limitations of) empirically-based arguments as well as their understandings that a general argument (i.e., a proof) treats the general case. In particular, we address the following two questions: To what extent do students think that examples suffice as proof? and To what extent do students recognize that a proof treats the general case?

Methods

Data were collected from 394 middle school students (grades 6-8); the students all attended the same middle school. The middle school recently adopted the reform-based curriculum *Connected Mathematics Program*; the adoption of this particular curricular program is noteworthy given the program's emphasis on mathematical reasoning (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002). The primary source of data was student responses to written assessment items. The two particular items that are the focus of this paper targeted the idea of generality. In the first item, students were given two arguments—one examples-based and one general (i.e., proof)—justifying the truth of a statement. Students were then asked to decide (and explain their decision) which argument demonstrated that the statement was always true. In the second item, students were given a statement and asked if the statement was true for a small set of numbers. In this latter item, students could either attempt to construct an argument demonstrating the statement's truth in general, or they could use the method of proof-by-exhaustion to demonstrate its truth for the specified set of numbers. A follow-up question then asked students if their justification also demonstrated that the statement was true for any number (not just those numbers included in the initial set). The students' responses to the assessment items were analyzed in terms of the two aspects of generality described previously. In particular, students' responses were coded using the following general coding descriptions (cf. Knuth, in progress; Waring, 2000): students consider checking a few cases as sufficient; students are aware that checking a few cases is not sufficient, but do not seem aware of the need for a general argument; students are aware of the need for a general argument, but perceive general arguments as limited (e.g., examples still need to be verified); students are aware that a general argument treats the general case.

Results & Discussion

Due to the page length limitations of the conference proceedings, results from the two focus items are briefly presented and discussed here (more detail as well as additional results will be presented during the conference session).

Assessment Item 1

Prior to presenting the results for this item, it is worth noting a relevant finding from the previous year's assessment: consistent with findings from previous research, students demonstrated an overwhelming reliance on the use examples as a means of demonstrating and/or verifying the truth of a statement. For example, the majority of students at all three grade levels "proved" that the sum of any two consecutive numbers is always an odd number by providing several examples demonstrating that the statement was indeed true; very few students attempted to provide a general argument. Students' reliance on the use of examples led us to wonder whether students actually believe that examples suffice as proof or whether it may be the case that they are aware of the limitation of empirical evidence, yet, are unable to produce a general argument themselves and thus examples-based arguments are their only recourse in attempting to justify (Healy & Hoyles, 2000, makes a similar observation). To address this issue we presented students with the following statement: *When you add any two consecutive numbers, the answer is always odd*. Students were then given two arguments justifying the statement (the arguments were attributed to fictitious students). The first argument, Samari's, shows three examples of consecutive numbers adding up to an odd sum, and a concluding statement that the given statement is true for all consecutive numbers because it is true for the three examples. The second argument, Ellen's, uses a deductive chain that begins by stating that with two consecutive numbers you always get one odd and one even number, and since an odd number and an even number sum to an odd number, any two consecutive numbers will always sum to an odd number. Following presentation of the two arguments, students were asked the following question: *Whose response tells us that if we were to add any two consecutive numbers we would get an answer that is an odd number? Explain your answer*.

Given that the majority of students produced examples-based justifications when asked to prove the statement on a previous assessment, one might conjecture that such a justification would also be the most popular choice among the students. To some extent this was indeed the case; across all three grades, approximately 40% of the students selected Samari's argument. Typical student explanations included:

> *Samari's response because it gives examples of 2 very different numbers, and it explains very well* (7th grade student).
>
> *Samari's. Because she explains it and she also gives examples to prove it* (7th grade student).
>
> *Samari's response because she actually has an answer to give that proves this. So you add two consecutive numbers together you will get an odd number* (8th grade student).
>
> *Samari's is correct because she can give proof and Ellen's just tells* (6th grade student).

A significant proportion of students (~30%), however, selected Ellen's argument as the argument that proves the statement. The following responses are representative:

> *Ellen's response makes more sense because Samari's response worked for those two numbers but it doesn't prove it always would* (6th grade student).
>
> *Ellen's because she tells why it will always be an odd number and Samari's show some examples that show some consecutive numbers and their answers* (6th grade student).
>
> *Ellen's because she tells us numbers go even, odd, even, odd, etc., and that when you add an even number with an odd number, the answer is always odd which Samari doesn't tell us, she just gives examples* (7th grade student).

> *Ellen's. Samari's response proves it's true for 2 pairs of numbers only. Ellen's proves it's true in all cases* (8th grade student).

The results suggest that when given the choice between a general argument and an examples-based argument, a significant proportion of students selected the general argument as the one that demonstrates the truth of the given statement for all cases. Thus, it may be that although many students are unable to produce general arguments themselves, they do seem to recognize the difference between a general argument and an examples-based argument and, moreover, they may view the general argument as a proof.

Assessment Item 2

In a written assessment presented to students during the previous school year, they responded to the following item (cf. Porteous, 1990): *Sarah discovers a cool number trick. She thinks of a number between 1 and 10, she adds 3 to the number, doubles the result, and then she writes this answer down. She goes back to the number she first thought of, she doubles it, she adds 6 to the result, and then she writes this answer down. [The preceding text was also accompanied by a worked out example using the number 7.] Will Sarah's two answers always be equal to each other for any number between 1 and 10?* Although the majority of students "proved" that Sarah's two answers would always be equal to each other by using examples, a significant proportion (~20%) used the method of proof-by-exhaustion to justify that the two answers would always be equal to each other. The students' use of this method prompted us to question whether students were knowingly (in a mathematical sense) exhausting the set of possibilities or whether they were simply testing examples (albeit the complete set). In other words, did these students perceive a difference between checking some cases and checking all cases in justifying the truth of a proposition? To address this question, we presented students with the same item the next year and, in addition, included the follow-up question: *Does your explanation show that the two answers will always be equal to each other for any number (not just numbers between 1 and 10)?*

Students' responses to the first part of the question were similar to the results from the previous year; however, their responses to the second part were perhaps the most interesting. Not surprisingly, the majority of students used examples as their method of justification for both parts. The following are representative of the responses (for both parts) from such students:

> *Yes because I tried some of the other numbers and for all of them I got the same answers. It applies for all numbers because I tried it with different examples [student shows two examples greater than 10]* (8th grade student).

> *Yes, the answers will always be equal because I tried her method using 5 and the results came out equal. I also tried 8 and the answers came out equal. Yes, it is true with every number because I tried that method with 11 and the answers came out equal* (6th grade student).

In contrast, some students seemed to recognize the limitation of examples as a means of proof:

> *[Student correctly works out an example using 8.] No, just because you do two examples [i.e., the given example and the student's worked out example] doesn't mean that if you*

do another two that they'll be the same. No, because if you do 3 more problems like these it doesn't mean that they will be equal to one another (6th grade student).

Somewhat similar in nature were responses from students who seemed to recognize that they could use proof-by-exhaustion to justify that the two answers would always be equal when the choice of numbers was limited to those between 1 and 10, and that this method of justification would not suffice for justifying that the two answers would be equal for numbers outside that range. The following responses are representative:

> *Yes, all results will be equal between 1 and 10 [student shows examples for all numbers 1-10, except 7 which was worked out as a part of the item]. No, my explanation shows the two answers will be equal just for numbers 1-10* (8th grade student).

> *Yes [student shows examples for numbers 1-10]. No, I only gave explanations for 1-10* (7th grade student).

These latter responses seem to suggest that for some students examples do not suffice as proof, and that they do in fact recognize the limitation of empirical evidence as a means of justification.

Interestingly, a number of students, who also used the method of proof-by-exhaustion for the first part, concluded (sans a general argument) that the two answers would be equal for all numbers. In some cases, students simply stated that because they tested the numbers 1-10 and the two answers were always equal, then the number trick would also work for numbers outside that range. For example, one student responded:

> *Yes [student shows examples for all numbers 1-10]. Yes, because I did it for each number between 1 and 10* (8th grade student).

In other cases, students based their justification for the second part on further examples:

> *Yes, Sarah's two answers will always be equal to each other for any numbers between 1 and 10 because I tried every number between 1 and 10 and it does work. My explanation shows that the two answers will always be equal to each other for any number not just numbers between 1 and 10 because if you tried 56+3=59; 59x2=118. 56x2=112; 112+6=118* (7th grade student).

> *Yes, Sarah's two answers will always be equal for 1-10. I know that because I tried each number 1-10. Yes, the two answers will always work for any number. I checked by trying different numbers, both large and small, odd and even* (6th grade student).

> *Yes, because I did examples for 1-10 numbers [student shows examples for numbers 1-10]. Yes, it does because if you do 12 or even 15, it will equal the same number* (8th grade student).

These latter two sets of responses suggest that these students may not perceive a difference between using proof-by-exhaustion as a method of *proof* and simply using examples as a method of justification—these students may have simply tested 10 examples and then tested additional examples to "widen" their examples-based justification.

Lastly, the data also include responses from students who demonstrated the ability to produce a general argument as their means of justification. In such cases, these students recognized the underlying mathematical relationship, although their articulation of the relationship varied in terms of clarity; nevertheless, it is clear that these students were attempting to treat the general case. Sample responses include:

> *Yes they will be equal because when you add 3 and multiply by 2 it's the same as multiplying by 2 and adding 6. Yes it does because: (a+3)2 = 2a+6. They are the same thing and ANY number will work* (8th grade student).

> *Yes, Sarah's answers will always equal to each other for any number between 1 and 10 because both number tricks are doing the same thing to the number. Both number tricks double the number. Since Sarah adds three before she doubles the number, she has to add six to the other trick because she doubles before she adds. [Student shows examples using 23 and 83 as the starting numbers.] Yes, this number trick will work for any number because each of the tricks is doing the same thing to the number, just written differently* (8th grade student).

There were also some interesting responses in which students presented a general argument for the first part, but then seemed to feel that the general argument was not "general." For example, one student felt that his general argument only applied for the numbers 1-10:

> *Yes, because if she always adds 3, then doubles your answer and gets that answer. And then goes back and does it in a different sequence she'll always get the same answer. No, because she only did numbers between 1 and 10* (6th grade student).

Another student responded similarly initially, but then had a change of mind—she seemed to realize that the general argument she presented was indeed general:

> *Yes, I think it would work because each time you get the answer you will get the same answer as the one before. I think this because the 1st time through, you add 3 and then times by 2. But the 2nd time through, you times by 2 and then add 6. The 2nd time through since you don't add 3 when you add 6, that is what doubles the 3 like the 1st time through, so both ways you add 6. No, it does not show that numbers higher than 10 would work. [The student then seems to change her mind.] As I showed above, it [the justification] is the same thing that would be used here* (6th grade student).

Finally, a number of students who presented a general argument felt the need to "prove" that their general argument was general by demonstrating with examples that it worked. The following student's response is representative:

> *The two answers will be equal because you're doing it [the steps] in reverse order and the reason there is a 6 instead of a 3 is because the 3 gets doubled in the second way.*

Yes, it does and to prove it I will show you [students shows that the number trick works using the number 17] (6th grade student).

Concluding Remarks

In closing, the results suggest that many middle school students lack an understanding of generality. Yet, the results also suggest that some students do possess an understanding of generality: Students produced and selected general arguments, recognized the limitation of examples as proof, and correctly used proof-by-exhaustion. If more students are to develop their understanding of generality—and of proving more specifically—then they must be given opportunities to engage in activities which highlight important ideas about proving.

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