MAPPING THE CONCEPTUAL TERRAIN OF MIDDLE SCHOOL STUDENTS' COMPETENCIES IN JUSTIFYING AND PROVING

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This paper presents preliminary results from the first year of a multi-year research project[i] exploring the development of middle school students' competencies in justifying and proving and the conditions and pedagogy necessary to promote that development. In particular, in this paper we present and discuss results from a written assessment completed by approximately 350 6th through 8th grade students. The assessment focused on students' abilities to both evaluate and generate mathematical arguments, their understanding of various aspects of deduction, and their understanding of the nature of proof. Consistent with previous results, students demonstrated an overwhelming reliance on the use examples as a means of demonstrating and/or verifying the truth of a statement. The results also suggest that there are differences among students (by grade level) with regard to their understanding of various aspects of the deductive process.

Many consider proof to be central to the discipline of mathematics and the practice of mathematicians. Yet surprisingly, the role of proof in school mathematics has traditionally been peripheral at best, usually limited to the domain of high school geometry. According to Wu (1996), however, the scarcity of proof in school mathematics is a misrepresentation of the nature of proof in mathematics: "Even as anomalies in education go, this is certainly more anomalous than others inasmuch as it presents a totally falsified picture of mathematics itself" (p. 228). Similarly, Schoenfeld (1994) claimed that "proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels" (p. 76). Sowder and Harel (1998) also argued against limiting students' experiences with proof to geometry, but more from an educational rather than mathematical perspective: "It seems clear that to delay exposure to reason-giving until the secondary-school geometry course and to expect at that point an instant appreciation for the more sophisticated mathematical justifications is an unreasonable expectation" (p. 674).

Reflecting an awareness of such criticism, as well as embracing the important role of proof in mathematical practice, recent reform efforts are calling for substantial changes in school mathematics with respect to proof. In contrast to the status of proof in the previous national standards document (National Council of Teachers of Mathematics [NCTM], 1989), its position has been significantly elevated in the most recent document (NCTM, 2000). In fact, not only has proof been upgraded to an actual standard in this latter document, but it has also received a much more prominent role throughout the *entire* school mathematics curriculum and is expected to be a part of the mathematics education of *all* students. More specifically, the *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that the mathematics education of pre-kindergarten through grade 12 students enable all students "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56).

These recommendations, however, pose serious challenges for school mathematics students given that many students have, traditionally, found the study of proof difficult. In fact, research has painted a bleak picture of students' competencies in justifying and proving (e.g., Bell, 1976; Chazan, 1993; Healy & Hoyles, 2000; Porteous, 1990; Senk, 1985; Usiskin, 1987). Although such research has made significant contributions to our understanding of students' competencies in justifying and proving, such research also has its limitations. As a collective, the studies do not provide a coherent picture of students' competencies over typical grade spans (e.g., middle school, high school), since each study was framed differently, studied different student populations in different school contexts, and relied on different instrumentation and methodology. Thus the goal of this session is to present and discuss results concerning the range of middle grade (6th - 8th) students' understandings of justification and proof—in short, to provide a map of the conceptual terrain of middle school students' competencies in justifying and proving.

Theoretical Perspective

Researchers have hypothesized that the development of students' competencies in justifying and proving might follow a developmental progression; that is, students' understandings of mathematical justification are "likely to proceed from inductive toward deductive and toward greater generality" (Simon & Blume, 1996, p. 9). Waring (2000), building upon the work of other researchers (e.g., Balacheff, 1991; Bell, 1976; Fischbein, 1982; van Dormolen, 1977), proposed six levels of proof concept development:

Level 0: Students are ignorant of the need for, or existence of, proof.

Level 1: Students are aware of the notion of proof, but consider checking a few cases as sufficient (i.e., Balacheff's naive empiricism).

Level 2: Students are aware that checking a few cases is not sufficient, but are satisfied that either i) checking extreme cases (i.e., Balacheff's crucial experiment) or random cases is proof, or ii) use of a generic example forms a proof for a class of objects (i.e., Balacheff's generic example).

Level 3: Students are aware of the need for a general argument, but are unable to produce such arguments themselves; however, they are likely to be able to understand the generation of such an argument (for example, by a classmate). This also includes the ability to follow a short chain of deductive reasoning.

Level 4: Students are aware of the need for a general argument, are able to understand the generation of such an argument, and are able to produce such arguments themselves in a limited number of (familiar) contexts.

Level 5: Students are aware of the need for a general argument, are able to understand the generation of such an argument (including more formal arguments), and are able to produce such arguments themselves in a variety of contexts (both familiar and unfamiliar).

This framework provided a lens for interpreting our assessment data; however, our analysis of the assessment data also resulted in further delineation of the framework. As proposed, Waring's (2000) framework focuses primarily on students' judgments of proof and approaches to proving; our revision includes further elaboration of Level 3. In particular, we have extended Level 3 to include students' understandings of various concepts (e.g., definitions, necessary and sufficient conditions) viewed as prerequisite to being able to understand and produce deductive arguments.

Methods

Approximately 350 middle school (6^{th} - 8^{th} grade) students participated in this study. These students responded to a written assessment designed to measure their competencies in justifying and proving. In all, students responded to fifteen different items during a 45 minute class period; using matrix sampling, each individual student responded to eight of the fifteen items. The assessment items required students to evaluate various arguments (cf. Healy & Hoyles, 2000), generate their own arguments (e.g., Show that the sum of any two consecutive whole numbers is always odd.), demonstrate understanding of implication rules and deduction (e.g., Given the statement: If two even numbers are multiplied, then their product is even. Decide if the following statement is true or false and explain why: 286 is even, so it is the product of two even numbers.), apply definitions (e.g., Given a definition of a quadrilateral, determine whether a particular figure is an example of a quadrilateral), and discuss the nature of proof (e.g., What does it mean to prove something in mathematics?). In several cases, the assessment items were adapted from prior research studies. Data analyses were informed primarily by the theoretical perspective described above.

Results

Due to limitations of page length, results are presented from a subset of items (full results will be presented during the conference session). Representative excerpts from students' written responses are provided to illustrate particular findings.

Producing Mathematical Arguments

Consistent with previous research, students demonstrated an overwhelming reliance on the use examples as a means of demonstrating and/or verifying the truth of a statement (Level 1/Level 2). Further, students did not indicate in their responses any recognition of the limitation of empirical evidence as a means for establishing the truth of a statement. As an example, 70% of the student responses to the two assessment items that required them to construct an argument justifying the truth of a given statement were based on the use of examples. The following are the two aforementioned assessment items: (1) *The sum of two consecutive numbers is always an odd number. For example, 5+6=11 and 8+9=17. Show that the sum of any two consecutive numbers is always an odd number*; and (2) *Show* *that when you add any two even numbers, your answer is always even. Provide an explanation that would convince a classmate that the answer is always even.*

One the one hand, the majority of these students appear to believe that providing one or more unsystematically selected examples is sufficient to prove a statement (Level 1). The following is a representative justification to Item 2: "I would take 5 different [pairs of] even numbers and then answer them [i.e., find the sum] and if she/he does not believe me, I will keep on doing it." In this case, the student seems to suggest that any five examples provide sufficient proof that the statement is always true and, if this justification fails to convince a classmate, then the selection of more examples should suffice to provide the necessary evidence to change the classmate's mind. On the other hand, a smaller percentage of these students strategically selected different types of numbers (e.g., very large/small numbers) in their efforts to produce a convincing argument (Level 2). For example, one student responded to Item 2 by selecting five pairs of even numbers: three pairs of small even numbers, one pair with one small and one large even number, and one pair of two large even numbers (including one over 100 million). In this case, the student employs a variety of numbers and combinations to show that the statement is true for different classes of numbers, including "very large" numbers. Responses of this nature suggest a certain level of sensitivity to the need to provide evidence that the statement is true, regardless of the size of the number.

A minority of students attempted to produce more general arguments for the two assessment items, with varying degrees of success. As an example, one student responding to Item 1 wrote: "Consecutive numbers are <u>always</u> one odd and one even number no matter whether I started with one even or one odd. Sums of an even number and an odd number are always odd." In this case, the student uses a feature of consecutive integers and then proceeds to utilize a mathematical declaration ("Sums of an even number and an odd number are always odd.") of his own in presenting an argument that the initial proposition is true. Although the student has attempted to present a general argument, acceptance of his argument as proof is dependent upon the acceptance of his mathematical declaration as true (to an extent, his argument is somewhat circular).

Students' Application of Definitions

On items in which students had to apply a mathematical definition, younger students had a tendency to use information external to that provided in the definition and, as a consequence, the use of such information often interfered with their interpretation of and ability to correctly respond to the assessment item. In other words, there seems to be a tendency, especially strong among younger students, to bring intuitive and familiar notions of a concept to bear in applying a definition; this tendency, however, appears to abate as students progress further in middle school. As an example, one assessment item required students to consider an unconventional, but technically correct, interpretation of the following definition of a quadrilateral: *A quadrilateral is what you get if you take four points A, B, C, and D and join them with four straight lines*. The "unconventional, but technically correct, interpretation" of the definition consisted of a figure with two of its four sides formed by connecting non-consecutive vertices (produced by a hypothetical middle school student named Carla). The students were then asked if this figure was a quadrilateral according to the stated definition.

In Grade 6, about two-thirds of the responses referred to a more generic definition of quadrilateral, whereas in Grade 8, about two-thirds of the responses used the specific definition provided to decide if the figure was indeed a quadrilateral. In the former case, the following is a typical 6^{th} grade student response, one that appeals to a more generic definition: "No, it is not a quadrilateral because the sides cross." It seems clear that the student is referring to his or her own definition of a quadrilateral rather than to the specific definition provided with the assessment item. In the latter case, the following is a typical 8^{th} grade student response: "According to the definition, Carla has drawn a quadrilateral. In the definition, it says you find any 4 points and connect them with straight lines, not in what order you have to connect them."

Understanding Conditional Statements

The data suggest that students are far more likely to correctly interpret the conditional nature of a mathematical statement if they could readily visualize a counterexample. For example, students responded to an item consisting of two conditional statements of the form "If (A and B), then C." More specifically, students were asked to state whether each of the following statements were true or false and to explain their choice: (a) *A quadrilateral that has four equal sides <u>must</u> be a square*, and (b) *A quadrilateral that has four right angles <u>must</u> be a square*. In (a), it was apparent that, at all grade levels, the students' lack of knowledge (or recall) regarding the existence of rhombi interfered with their ability to generate a counter-example and, thus, their ability to correctly respond to the question. Interestingly, this lack of knowledge (or recall) was also evident in (b), however, more so with the 6^{th} grade students than the 8^{th} grade students (statistically significant, $p < 0.05$), that is, more 6^{th} grade students responded incorrectly than did 8^{th} grade students. Apparently, the 8^{th} grade students had a greater familiarity with the properties of rectangles than did the 6^{th} grade students—the 6^{th} grade students may not have viewed a square as a special case of rectangle—and as a consequence, the 8th grade students were better able to generate a counter-example. This result may be a curricular effect in that 6th grade students may not have yet learned to classify quadrilaterals.

Concluding Remarks

Previous research has suggested that many teachers have inadequate conceptions of proof (e.g., Harel & Sowder, 1998; Jones, 1997; Knuth, in press; Martin & Harel, 1989) and that they have limited views regarding the nature and role of proof in school mathematics (Knuth, 2002). Consequently, engaging teachers in discussions focused on the details of students' competencies in justifying and proving may provide a basis for enhancing both teachers' own understandings of proof and their perspectives regarding proof in school mathematics. In addition, such detail on student reasoning may also provide a basis for continued growth and development of teachers' understandings of their students' reasoning and, consequently, their abilities to support the development of their students' mathematical reasoning (cf. Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1996).

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