Middle School Students' Understanding of Core Algebraic Concepts: Equivalence & Variable[1]

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Abstract: Algebra is a focal point of reform efforts in mathematics education, with many mathematics educators advocating that algebraic reasoning should be integrated at all grade levels K-12. Recent research has begun to investigate algebra reform in the context of elementary school (grades K-5) mathematics, focusing in particular on the development of algebraic reasoning. Yet, to date, little research has focused on the development of algebraic reasoning in middle school (grades 6-8). This article focuses on middle school students' understanding of two core algebraic ideas—equivalence and variable—and the relationship of their understanding to performance on problems that require use of these two ideas. The data suggest that students' understanding of these core ideas influences their success in solving problems, the strategies they use in their solution processes, and the justifications they provide for their solutions. Implications for instruction and curricular design are discussed.

Kurzreferat: Algebra ist einer der Schwerpunkte der Reformbemühungen in der Mathematikdidaktik, und viele Mathematikdidaktiker fordern daher, dass algebraisches Begründen in alle Klassenstufen ab der Vorschule bis Jahrgangsstufe 12 integriert werden sollte. Neuere Studien haben begonnen, Reformen des Algebraunterrichts im Kontext des Mathematikunterrichts an der Grundschule zu untersuchen; dabei legen sie den Schwerpunkt auf die Entwicklung von algebraischem Argumentieren. Bisher gibt es nur wenige Untersuchungen, die sich mit der Entwicklung des algebraischen Argumentierens im unteren Sekundarbereich befassen. Dieser Beitrag untersucht nun vor allem das Verständnis von Lernenden des unteren Sekundarbereichs hinsichtlich zweier algebraischer Grundideen - Gleichungen und Variablen - sowie den Zusammenhang ihres Verständnisses mit ihren Leistungen bei Aufgaben, für welche die Anwendung dieser beiden Grundideen zentral ist. Die Daten legen die These nahe, dass das Verständnis der Lernenden hinsichtlich dieser beiden zentralen Ideen sowohl Auswirkungen auf ihre Erfolge beim Lösen der Aufgaben haben, als auch auf die während des Lösungsprozesses verwendeten Strategien sowie auf die von ihnen gegebenen Begründungen für ihre Lösungen. Ferner werden Auswirkungen auf Unterricht und Curriculumentwicklung diskutiert.

ZDM-Classification: C30, H20

1. Introduction

Algebra is considered by many to be a "gatekeeper" in school mathematics, critical to further study in mathematics as well as to future educational and employment opportunities (Ladson-Billings 1998; National Research Council [NRC] 1998). Unfortunately, many students experience difficulty learning algebra (Kieran 1992), a fact that has led to first-year algebra courses in the United States being characterized as "an unmitigated disaster for most students". (NRC p. 1) In response to growing concern about students' inadequate understandings and preparation in algebra, and in recognition of the role algebra plays as a gatekeeper, recent reform efforts in mathematics education have made algebra curricula and instruction a focal point (e.g., Bednarz, Kieran, & Lee 1996; Lacampagne, Blair & Kaput 1995; National Council of Teachers of Mathematics 1997, 2000; NRC 1998; RAND Mathematics Study Panel 2003). In fact, Kaput (1998) has argued that algebra is the keystone of mathematics reform, and that teachers' abilities to facilitate the development of students' algebraic reasoning is the most critical factor in algebra reform. Moreover, he contended that the "key to algebra reform is integrating algebraic reasoning across all grades and all topics—to 'algebrafy' school mathematics". (p. 1)

Underlying this call to 'algebrafy' school mathematics is a belief that the traditional separation of arithmetic and algebra deprives students of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades (Kieran 1992). Algebrafying school mathematics, however, means more than moving the traditional first-year algebra curriculum down to the lower grades. There is a growing consensus that algebra reform requires a reconceptualization of the nature of algebra and algebraic reasoning as well as a reexamination of when children are capable of reasoning algebraically and when ideas that require algebraic reasoning should be introduced into the curriculum (Carpenter & Levi 1999). Recent research has begun to investigate algebra reform in the context of elementary school mathematics, focusing in particular on the development of algebraic reasoning (e.g., Bastable & Schifter in press; Carpenter, Franke, & Levi 2003; Carpenter & Levi; Kaput in press). Yet, to date, little research has focused on the development of algebraic reasoning in the middle grades—the time period linking students' arithmetic and early algebraic reasoning and their development of increasingly complex, abstract algebraic reasoning. In this article, we present results from a multi-year research project that seeks to understand the development of middle school students' algebraic reasoning. In particular, the article focuses on students' understanding of two core algebraic ideas—equivalence and variable—and the relationship of their understanding to performance on problems that require use of these two ideas.

2. Student understanding of equivalence & variable

Algebraic reasoning depends on an understanding of a number of key ideas, of which equivalence and variable are, arguably, two of the most fundamental. In this section we briefly describe research that has examined

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students' understandings of these two ideas; this description will serve to situate the present study in the larger body of research as well as to highlight the contribution of the present study.

2.1 Equivalence

The ubiquitous presence of the equal sign symbol in mathematics at all levels highlights its important role in mathematics, in general, and in algebra, in particular. Within the domain of algebra, Kieran (1992) contended that "one of the requirements for generating and adequately interpreting structural representations such as equations is a conception of the symmetric and transitive character of equality—sometimes referred to as the 'left-right equivalence' of the equal sign". (p. 398) Yet, there is abundant literature that suggests students do not view the equal sign as a symbol of equivalence (i.e., a symbol that denotes a relationship between two quantities), but rather as an announcement of the result or answer of an arithmetic operation (e.g., Falkner, Levi, & Carpenter 1999; Kieran 1981; McNeil & Alibali in press; Rittle-Johnson & Alibali 1999). For example, Kieran (1981) found that 12- and 13-year old students described the equal sign in terms of the answer and provided examples of its use that included an operation on the left-hand side of the symbol and the result on the right-hand side (e.g., $3 + 4 = 7$). McNeil and Alibali (in press) found similar conceptions of the equal sign in definitions generated by third- through fifth-grade students.

While such a (mis)conception concerning the meaning of the equal sign may not be problematic in elementary school, where students are typically asked to solve equations of the form $a + b = \square$, it does not serve students well in terms of their preparation for algebra and algebraic ways of thinking. In algebra, students must view the equal sign as a relational symbol (i.e., "the same as") rather than as an operational symbol (i.e., "do something"). The relational view of the equal sign becomes particularly important as students encounter and learn to solve algebraic equations with operations on both sides of the symbol (e.g., $3x - 5 = 2x + 1$). A relational view of the equal sign is essential to understanding that the transformations performed in the process of solving an equation preserve the equivalence relation (i.e., the transformed equations are equivalent)—an idea that many students find difficult, and that is not an explicit focus of typical instruction. Steinberg, Sleeman, and Ktorza (1990) concluded that many eighth- and ninth-grade students do not have a good understanding of equivalent equations. They found that many students knew how to use transformations in solving equations, however, many of these same students did not seem to utilize such knowledge in determining whether two given equations were equivalent. Although not examined in their study, it seems reasonable to conclude that many of these latter students may have had inadequate conceptions of mathematical equivalence.

2.2 Variable

Algebra has been called the study of the 24th letter of the alphabet. Although this characterization is somewhat facetious, it underscores the importance of developing a meaningful conception of variable in learning and using algebra. The idea of variable, not surprisingly, has also received considerable attention in the mathematics educational research community (e.g., Küchemann 1978; MacGregor & Stacey 1997; Philipp 1992; Usiskin 1988), and the results of such work suggest that the use of literal symbols in algebra presents a difficult challenge for students. In Küchemann's frequently cited study, for example, he found that most 13-, 14-, and 15-year-old students considered literal symbols as objects (i.e., the literal symbol is interpreted as a label for an object or as an object itself). Few students considered them as specific unknowns (i.e., the literal symbol is interpreted as an unknown number with a fixed value), and fewer still as generalized numbers (i.e., the literal symbol is taken to represent multiple values, although it is only necessary to think of the symbol taking on these values one at a time) or variables (i.e., the literal symbol represents, at once, a range of numbers). Further, his study showed that students' misunderstandings of literal symbols seem to be reflected in their approaches to symbolizing relationships in problem solutions—an essential aspect of algebra and algebraic ways of thinking.

In sum, developing an understanding of equivalence and variable is essential to algebra and the ability to use it, yet they are ideas about which many students have inadequate understandings. In this article, we examine the meanings middle school students ascribe to the equal sign and variable, their performance on problems that require use of these ideas, and the relationship between the meanings they ascribe to each and their performance on the corresponding problems.

3. Method

3.1 Participants

Participants were 373 middle-school (6th through 8th grade) students drawn from an ethnically diverse middle school in the American Midwest. The demographic breakdown of the school's student population is as follows: 25% African American, 5% Hispanic, 7% Asian, and 62% White. The middle school had recently adopted a reform-based curricular program, *Connected Mathematics*, and, with the exception of one section of 8th grade algebra, the classes were not tracked (e.g., all 6th grade students were in the same mathematics course). The school was selected as the site for this research based upon the recommendation of the school district's mathematics resource teacher, who felt that the principal and teachers would be interested in participating.

3.2 Data collection

The data that are the focus of this article consist of students' responses to a subset of items from a written assessment that targeted their understandings of various aspects of algebra. In particular, the focus is on four items that were designed to assess students' understanding of the ideas of equal sign (1 item) and of variable (1 item) as well as their performance on two problem solving items that (potentially) required the use of these ideas. The assessment consisted of three forms with some overlap of items; all 373 students were administered the equal sign understanding and variable understanding items, 251 students were administered the equal sign performance item, and 122 students were administered the variable performance item. The assessment was administered near the beginning of the school year.

3.2.1 Equal sign items

In the first item (shown in Fig. 1), students were asked to define the equal sign. The rationale for the first prompt

The following questions are about this statement:

$3 + 4 = 7$

↑

a) The arrow above points to a symbol. What is the name of the symbol?
b) What does the symbol mean?
c) Can the symbol mean anything else? If yes, please explain.

Figure 1: Interpreting the equal sign.

(What is the name of the symbol?) was to preempt students from using the name of the symbol in their response to the second prompt (What does the symbol mean?). The rationale for the third prompt (Can the symbol mean anything else?) was to provide students the opportunity to give an alternative interpretation; in previous work, we have found that students often offer more than one interpretation when given the opportunity. The second item (shown in Fig. 2), the *equivalent equations* problem, was designed to assess students' understanding of the fact that the transformations

Is the number that goes in the □ the same number in the following two equations? Explain your reasoning.

$2 \times \square + 15 = 31$ $2 \times \square + 15 - 9 = 31 - 9$

Figure 2: Using the concept of mathematical equivalence.

performed in the process of solving an equation preserve the equivalence relation. We expected that students who viewed the equal sign as representing a relationship between quantities would conclude that the number that goes in the box is the same in both equations because the transformation performed on the second of the two equations preserved the quantitative relationship expressed in the first equation.[2]

3.2.2 Variable items

Item 3 (shown in Fig. 3) was designed to assess students' interpretations of literal symbols. The fourth item (shown in Fig. 4), the *which is larger* problem, was designed to assess students' abilities to use the concept of variable to make a judgment about two varying quantities. In particular, to be successful on the final item, students must recognize that the values of $3n$ and $n + 6$ are dynamic and depend on the value of n, that is, they must

The following question is about this expression:

$2n + 3$

↑

The arrow above points to a symbol. What does the symbol stand for?

Figure 3: Interpreting a literal symbol used as a variable.

Can you tell which is larger, $3n$ or $n + 6$? Please explain your answer.

Figure 4: Using the concept of variable.

view n as a variable—a literal symbol that represents, at once, a range of numbers.

3.3 Coding

In this section we provide details regarding the coding of each item; in the results section, we provide sample student responses. For all items, responses that students left blank or for which they wrote "I don't know" were grouped in a *no response/don't know* category, and responses for which students' reasoning could not be determined and responses that were not sufficiently frequent to warrant their own codes were grouped in an *other* category.

3.3.1 Coding equal sign understanding

Student responses to parts b) and c) of Item 1 were coded as *relational*, *operational*, *other*, or *no response/don't know*, with the majority of responses falling into the first two categories. A response was coded as *relational* if a student expressed the general idea that the equal sign means "the same as" and as *operational* if the student expressed the general idea that the equal sign means "add the numbers" or "the answer". In addition to coding the responses to parts b) and c) separately, students were also assigned an overall code indicating their "best" interpretation. Many students provided two interpretations, often one *relational* and one *operational*; in such cases, the responses were assigned an overall code of *relational*.

3.3.2 Coding performance on the equivalent equations *problem*

Students' responses to Item 2 were coded for correctness as well as strategy use. Responses were coded as correct if students responded that the two equations have the same solution. Students' strategies for solving the problem were classified into one of five categories: *answer after equal sign*, *recognize equivalence*, *solve and compare*, *other*, or *no response/don't know*. In the *answer after equal sign* category, students' rationale for their conclusion was that each equation had the same "answer" (in this case, 31) to the immediate right of the equal sign and the equations were therefore equivalent (an incorrect strategy). In the *solve and compare* category, students' rationale for their conclusion was based on either (1) determining the solution to the first equation, substituting that solution into the second equation, and noting that the

[2] There were two versions of the *equivalent equations* problem, one which used a box (as in Fig. 2) and one which used n to represent the missing values. Performance did not differ across versions (60% correct box, 61% correct n), and the distribution of strategies used to solve the problem did not differ across versions (χ^2 (4, N = 252) = 1.729, *ns*), so we collapse across versions in the analyses presented in this paper.

value satisfied both equations, or (2) determining the solutions to both equations and comparing them. Finally, in the *recognize equivalence* category, students' rationale for their conclusion was based on recognizing that the transformation performed on the second equation preserved the equivalence relation. Note that only the *recognize equivalence* strategy appears to explicitly require a relational understanding of the equal sign.

3.3.3 Coding variable understanding

Students' responses to the literal symbol interpretation item were classified into five categories, *multiple values*, *specific number*, *object*, *other*, or *no response/don't know*. A response was coded as *multiple values* if the student expressed the general idea that the literal symbol could represent more than one value; as *specific number* if the student indicated that the literal symbol represents a particular number; and as *object* if the student suggested that the literal symbol represents a label for a physical object (such as stating that *n* represents newspapers).

3.3.4 Coding performance on the which is larger *problem*

Students' responses to the *which is larger* problem were coded both in terms of the judgment about which quantity was larger ($3n$, $n + 6$, or can't tell) and for the reasoning underlying that judgment. Students' explanations of their reasoning were classified into five categories: *variable* explanations, *single-value* explanations, *operation* explanations, *other*, or *no response/don't know. Variable* explanations expressed the general idea that one cannot determine which quantity is larger because the variable can take on multiple values. *Single-value* explanations tested a single value and drew a conclusion on that basis; thus, students' conclusions varied depending on the value tested. *Operation* explanations expressed the general idea that one type of operation leads to larger values than the other (for example, multiplication produces larger values than addition).

3.3.5 Coding reliability

To assess reliability of the coding procedures, a second coder rescored approximately 20% of the data. Agreement between coders was 90% for coding students' interpretations of the equal sign, 91% for coding students' strategies on the *equivalent equations* problem, 91% for coding students' interpretations of literal symbols, and 95% for coding students' explanations on the *which is larger* problem.

4. Results

We focus first on students' interpretations of the equal sign symbol, and how these interpretations relate to performance on the *equivalent equations* problem. We then turn to students' interpretations of a literal symbol (n) used as a variable, and how these interpretations relate to performance on the *which is larger* problem. Representative excerpts from students' written responses are provided to illustrate particular findings. In reporting the results, we describe (and illustrate) only those coding categories that are most germane to the focus of the article. Finally, the statistical analysis of the data was performed using logistic regression because the outcome variables of interest were categorical. All reported statistics are significant with alpha set at .05.

4.1 Interpretation of the equal sign

We first examined the relationship between grade level and interpretation of the equal sign symbol. The following responses are typical of those coded as *operational*:

"It means the total of the numbers before it". (6^{th} grade student)

"It means whatever is after it is the answer". (8^{th} grade student)

The following responses are typical of those coded as *relational*:

"It means the number(s) on its left are equivalent to the number(s) on its right". (6^{th} grade student)

"The things on both sides of it are of the same value". (7^{th} grade student)

Students were classified as providing a relational interpretation if they provided one on either their first or second response. As seen in Fig. 5, the proportion of

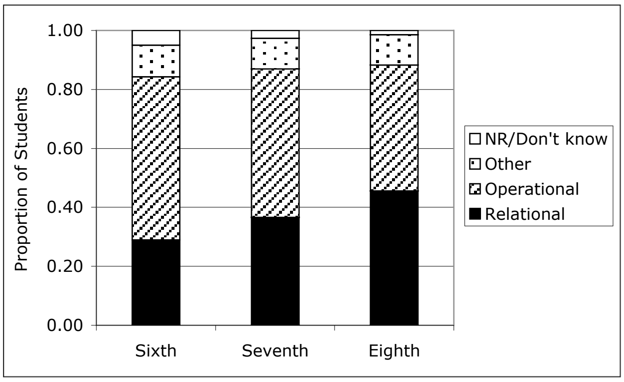

Figure 5: Equal sign interpretations of sixth-, seventh-, and eighth-grade students.

students providing a relational interpretation for the equal sign differed across the grades, *Wald* (2, N = 373) = 7.80, and this difference was accounted for by a significant linear trend, $\hat{\beta} = 0.52$, $z = 2.78$, *Wald* (1, N = 373) = 7.72. Despite this improvement across grades, however, the overall level of performance was strikingly low. Even at grade 8, only 46% of students provided a relational interpretation of the equal sign.

4.2 Performance on the* equivalent equations *problem

The proportion of students who correctly judged that the two equations had the same solution differed across the grades, *Wald* (2, N = 251) = 10.21, p = .006, and was accounted for by a significant linear trend, $\hat{\beta} = 0.72$, $z = 3.18$, *Wald* (1, N = 251) = 10.08, p = .002. Students' strategies for solving the *equivalent equations* problem are displayed in Table 1. The majority of students' strategies were categorized into one of the following categories: *recognize equivalence*, *solve and compare*, or *answer after equal sign*. Typical responses in each of these three categories included the following:

"Yes because you're doing the same equation but just minusing 9 from both sides in the second one". (*recognize equivalence*, 8th grade student)

"Yes if you substitute 8 for n, each answer will be equal and make sense [student shows computations for determining the value of *n* and for checking that the value satisfies the second equation]". (*solve and compare*, 6th grade student)

"Yes because it has to be to get a 31 in both answers". (*answer after equal sign*, 6th grade student)

As seen in Table 1, there was also a substantial number of students who left their answer sheets blank, simply wrote that they did not know, or used idiosyncratic strategies (i.e., strategies that could not be determined or that were insufficiently frequent to warrant their own codes, both of which were classified as *other* strategies). To some extent, the large proportion of strategies in the *other* category may not be too surprising: with the exception of one 8th grade algebra class, the students' exposure to algebra, in general, and equivalent equations, in particular, had been minimal at best. (Alternatively, it is encouraging to see that so many students—students who used the *recognize equivalence* or *solve and compare* strategies—were able to engage with the problem in mathematically appropriate ways prior to formal instruction in "algebra".)

Table 1: Proportion of students at each grade level who used each strategy use on the *equivalent equations* problem.

Strategy	Grade Level 6th	7th	8th
Recognize equivalence	0.12	0.17	0.34
Solve and compare	0.39	0.33	0.25
Answer after equal sign	0.11	0.11	0.11
Other	0.31	0.25	0.27
No response/Don't know	0.08	0.14	0.02

Is interpretation of the equal sign associated with performance on problems that involve equations? More specifically, do students who hold a relational view of this symbol perform better than their peers who do not hold such a view on a problem for which they must judge the equivalence of two equations? To find out, we examined the relationships among grade level (6, 7 or 8), equal sign interpretation (relational or not), and performance on the *equivalent equations* problem. We first consider students' judgments about whether the two equations had the same solutions or not, and then we consider their strategies for arriving at those judgments.

As seen in Fig. 6, students who provided a relational interpretation of the equal sign were more likely to judge that the two equations had the same solutions than were students who did not provide a relational interpretation. The effect of equal sign interpretation was significant when controlling for grade level, $\hat{\beta} = -1.24$, $z = -4.15$, *Wald* (1, N = 251) = 17.23. In addition, the effect of grade level was significant when controlling for equal sign interpretation, *Wald* (2, N = 251) = 9.00, and was accounted for by a significant linear trend, $\hat{\beta} = 0.70$, $z = 2.98$, *Wald* (1, N = 251) = 8.89.

As seen in Fig. 7, students who provided a relational interpretation for the equal sign were also more likely to use the *recognize equivalence* strategy than were students who did not provide a relational interpretation. The effect of equal sign interpretation was significant when controlling for grade, $\hat{\beta} = -1.33$, $z = -4.01$, *Wald* (1, N = 251) = 16.10. In addition, the effect of grade level was significant when controlling for equal sign interpretation, *Wald* (2, N = 251) = 12.11, and was accounted for by a significant linear trend, $\hat{\beta} = 0.93$, $z = 3.17$, *Wald* (1, N = 251) = 10.05. It is worth noting that a subset of students who used the recognize equivalence strategy (24%) displayed an operational view of equality on the equal sign interpretation item. Thus, different problem contexts appear to activate or draw on different aspects of students' knowledge.

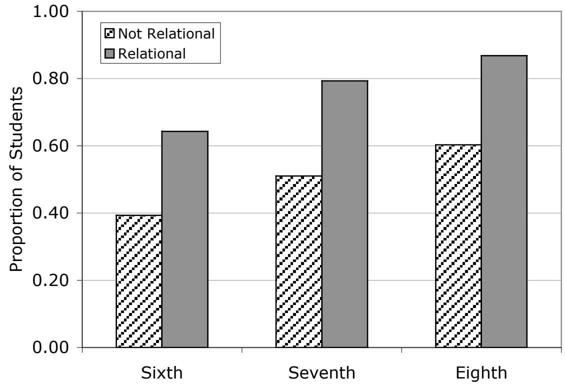

Figure 6: Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category who answered the *equivalent equations* problem correctly.

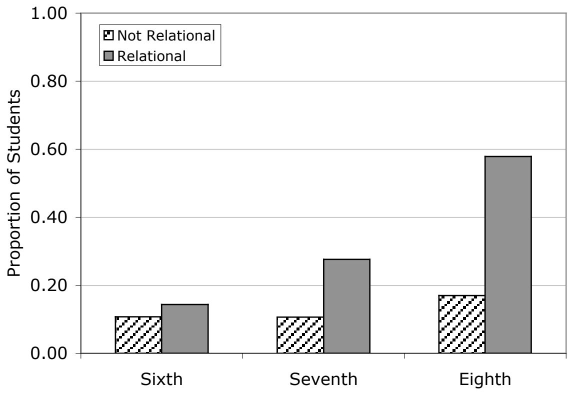

Figure 7: Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category who used the *recognize equivalence* strategy on the *equivalent equations* problem.

In sum, students' understanding of the equal sign was associated with their performance on the *equivalent equations* problem, both in terms of their judgments for the problem and the strategies they used to arrive at those judgments. Thus, students who demonstrated a relational understanding of the equal sign appeared to use this understanding in determining that the two equations had the same solutions.

4.3 Interpretation of a literal symbol

We turn now to students' interpretations of a literal symbol (n) used as a variable in a mathematical expression. The most common meaning students at all three grade levels provided was that of a variable—the literal symbol could represent more than one value. The following responses are representative of the *multiple values* code:

"The symbol is a variable, it can stand for anything". (6th grade student)

"A number, it could be 7, 59, or even 363.0285". (7th grade student)

"That symbol stands for x which stands for a number that goes there". (8th grade student)

In the final example above, it is interesting that the student apparently felt the need to replace n with x, the latter of which represents a number; this response may be an artifact of school mathematics in which the prototypical literal symbol is x. Not surprisingly, as seen in Fig. 8, the proportion of students providing a correct

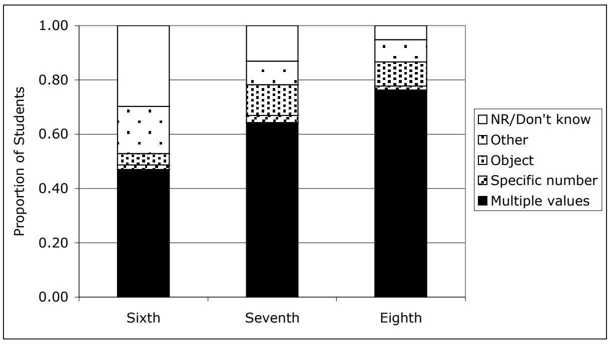

Figure 8: Students' interpretations of a literal symbol.

interpretation (i.e., multiple values) differed across the grades, *Wald* (2, N = 372) = 22.58, increasing from fewer than 50% of students in grade 6 to more than 75% of students in grade 8. This improvement across grades was accounted for by a significant linear trend, $\hat{\beta} = 0.91$, $z = 4.71$, *Wald* (1, N = 372) = 22.27.

It is also worth noting the relatively large proportion of 6th grade students whose responses were categorized as either *other* or *no response/don't know*. One possible explanation for the nature of these students' responses relates again to the curriculum: the first formal introduction of the concept of variable does not occur until the 7th grade (in the *Connected Mathematics* curriculum), thus the 6th grade students may lack experience with literal symbols used as variables in algebraic expressions.

4.4 Performance on the **which is larger** *problem*

Fig. 9 displays students' judgments to the question prompt (i.e., Can you tell which is larger, $3n$ or $n + 6$?), and Table 2 displays the justifications students provided for their judgments. In some cases, students provided only a judgment and not a justification for their judgment (justifications in these cases were assigned to the *no response/don't know* category).

In the 6th grade, the majority of students appeared either unable to provide a justification or to provide an idiosyncratic justification (see Table 2). Relative to the 6th grade students, the 7th and 8th grade students were more likely to respond with a correct justification that focused on the fact that the literal symbol could take on multiple

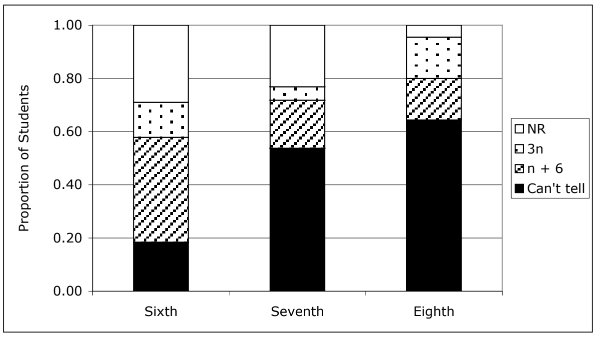

Figure 9: Students' judgments on the *which is larger* problem.

values. The following justifications are typical of *variable* responses:

"No because you don't know what n is". (6th grade student)

"No, because n is not a definite number. If n was 1, $3n$ would be 3 and $n + 6$ would be 7, but if n was 100, $3n$ would be 300 and $n + 6$ would be 106. This proves that you cannot tell which is larger unless you know the value of n". (8th grade student)

Although the coding category of *single value* appeared in fewer than 5% of the responses at each grade level, it is worth noting, because these students at least seemed to

Table 2: Proportion of students at each grade level who provided each type of justification for the *which is larger* problem.

Justification	Grade Level 6th	7th	8th
Variable	0.11	0.51	0.60
Single Value	0.03	0.05	0.04
Operation	0.00	0.05	0.09
Other	0.42	0.15	0.16
No response/Don't know	0.45	0.23	0.11

recognize that the literal symbol represents a number. In such cases, the students tested a specific number and based their judgments on the results of their computations. Likewise, the coding category of *operation* was also rare. Based on prior work (e.g., Greer 1992), we expected that some students would focus on the operation (for example, one seventh-grade student stated, "Yes, $n + 6$ is bigger because they +".). Although such responses did occur in 7th and 8th grades, they represented fewer than 10% of the responses at each grade level.

Is holding a multiple-values interpretation of n associated with performance on the *which is larger* problem? That is, were students who interpreted the literal symbol (item 3) as a variable more likely to answer "can't tell" and to provide a correct justification than were students who did not provide a multiple-values interpretation? To find out, we examined the relationships among grade level (6, 7 or 8), literal symbol interpretation (multiple values or not), and performance on the *which is larger* problem. We first consider students' judgments about whether $3n$ or $n + 6$ is larger, and then we consider their justifications.

As seen in Fig. 10, students who provided a multiple-values interpretation were indeed more likely than their peers who did not provide a multiple-values interpretation to answer "can't tell" on the *which is larger* problem. The effect of having a multiple-values interpretation was significant when controlling for grade level, $\hat{\beta} = -0.97$, $z = 2.22$, $Wald\ (1, N = 122) = 4.90$. In addition, the proportion of students who correctly answered "can't tell" increased across the grade levels. The effect of grade level on performance was significant when controlling for literal symbol interpretation, $Wald\ (2, N = 122) = 11.54$, and was accounted for by a significant linear trend, $\hat{\beta} = 1.27$, $z = 3.34$, $Wald\ (1, N = 122) = 11.13$.

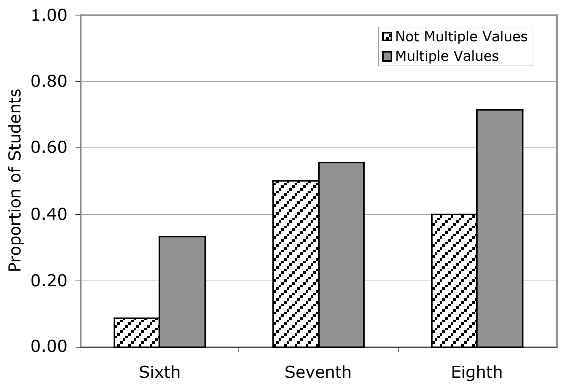

Figure 10: Proportion of sixth-, seventh-, and eighth-grade students in each literal symbol interpretation category who provided a correct judgment for the *which is larger* problem.

Lastly, as seen in Fig. 11, students who provided a multiple-values interpretation of the literal symbol were also more likely than were students who did not provide a multiple-values interpretation to provide correct

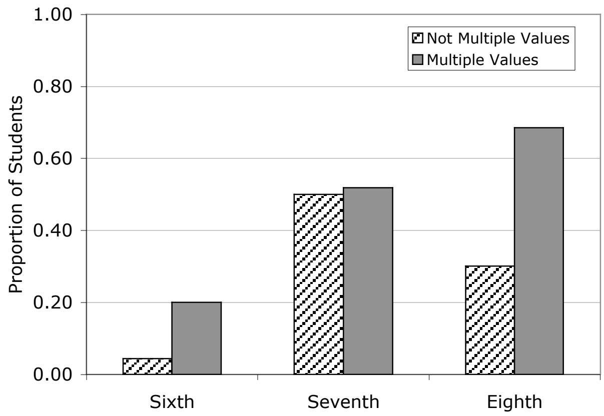

Figure 11: Proportion of sixth-, seventh-, and eighth-grade students in each literal symbol interpretation category who provided a correct justification for the *which is larger* problem.

justifications on the *which is larger* problem. The effect of having a multiple-values interpretation was significant when controlling for grade, $\hat{\beta} = -0.95$, $z = -2.05$, $Wald\ (1, N = 122) = 4.21$. Further, the proportion of students who provided a correct justification increased across the grade levels. The overall effect of grade was significant when controlling for literal symbol interpretation, $Wald\ (2, N = 122) = 13.79$, and was accounted for by a significant linear trend, $\hat{\beta} = 1.61$, $z = 3.65$, $Wald\ (1, N = 122) = 13.32$. It is worth noting that a subset of students who provided a correct justification on the *which is larger* item (20%) did not provide a multiple values interpretation on the variable understanding item. Thus, as for the equal sign items, different problem contexts appear to activate or draw on different aspects of students' knowledge.

In sum, understanding of variable was associated with performance on the *which is larger* problem, in terms of both students' judgments about which quantity was larger and their justifications for their judgments. Thus, students who had a multiple-values interpretation of a literal symbol appeared to use this understanding in determining that one cannot tell whether $3n$ or $n + 6$ is larger.

5. Discussion

The focus of this paper was on middle school students' understandings of the ideas of equivalence and variable, their performance on problems that require use of these ideas, and the relationship of their understanding to performance. In this section, we briefly discuss the results and their implications for mathematics education.

5.1 Equivalence Results

The finding that many students hold an operational view of the equal sign is not particularly surprising, given that similar results have been found in previous research (e.g., Falkner et al., 1999; Kieran 1981; McNeil & Alibali in press; Rittle-Johnson & Alibali 1999). Although our results suggest that students' views of the symbol become more mathematically sophisticated (i.e., view the equal sign as a relation between two quantities) as they progress through middle school, the majority of students at each grade level continued to exhibit less sophisticated views of the equal sign (e.g., as a "do something" symbol). This result is troublesome in light of our finding that students who have a relational view of the equal sign outperformed their peers who hold alternative views on a problem that requires use of the idea of mathematical equivalence. We report elsewhere that middle school students' views of the equal sign also play a role in their success in solving algebraic equations and simple algebra word problems (Knuth, Stephens, McNeil, & Alibali under review). Taken together, such results suggest that an understanding of equivalence is a pivotal aspect of algebraic reasoning and development. Consequently, students' preparation for and eventual success in algebra may be dependent on efforts to enhance their understanding of mathematical equivalence and the meaning of the equal sign.

Yet, equivalence is a concept traditionally introduced during students' early elementary school education, with little instructional time explicitly spent on the concept in the later grades. In fact, teachers generally assume that once students have been introduced to the concept during their elementary school education, little or no review is needed. Some previous work at the elementary school level has focused on promoting a relational view of the equal sign (e.g, Carpenter et al. 2003); however, there is little explicit attention to this concept in the later grades. This lack of attention may explain, in large part, why many students continue to show inadequate understandings of the meaning of the equal sign in secondary school and even into college (e.g., McNeil & Alibali in press; Mevarech & Yitschak 1983). Further exacerbating students' opportunities to develop their understanding of equivalence is the fact that very little attention is paid to the concept in curricular materials—despite the ubiquitous presence of the equal sign. Moreover, analyses of middle school curricular materials suggest that relational uses of the equal sign are less common than operational uses (McNeil, Grandau, Stephens, Krill, Alibali, & Knuth 2004). This pattern of exposure may actually condition students to favor less sophisticated and generalized uses of equivalence (such as "operations equals answer").

5.2 Variable Results

The findings regarding students' views of literal symbols are, in general, more positive than the results of previous research (cf. Küchemann 1978). In particular, a substantial proportion of students interpreted a literal symbol as representing more than one value, increasing from approximately 50% of the 6th grade students to more than 75% of the 8th grade students. Students' use of this knowledge suggests, however, that knowledge of the concept of variable may be somewhat fragile, particularly among 6th-grade students, who were largely unable to correctly answer the *which is larger* problem. Yet, those students who did provide a multiple-values interpretation of a literal symbol were more likely than their peers to not only use this understanding to determine that one could not tell whether $3n$ or $n + 6$ is larger, but also to provide a correct justification for why one could not tell which is larger. These latter results highlight the importance of fostering a multiple-values interpretation of literal symbols and suggest that efforts to foster such an interpretation will likely contribute to students' preparation for algebra and algebraic ways of thinking.

In contrast to the treatment of equivalence, the concept of variable is one that receives explicit instructional and curricular attention in middle school (7th grade in the *Connected Mathematics* curriculum). It may be the case, however, that providing students with opportunities to meaningfully encounter literal symbols in ways that support the development of a multiple-values understanding at an earlier age may be beneficial in terms of their preparation for and eventual success in algebra (a perspective shared by others, e.g., Carraher, Brizuela, & Schliemann 2000). Students often encounter literal symbols during their elementary school education (e.g., $8 + 3 = \square$, $3 + ? = 7$), however, the nature of such exposure may lead students to consider literal symbols in less sophisticated and mathematically powerful ways (e.g., as specific numbers).

6. Concluding remarks

If a goal of mathematics education reform is to better prepare *all* students for success in algebra, then the nature of students' "pre-algebraic" mathematical experiences must lay the foundation for more formal study of algebra. Much of this foundation can be laid as well as strengthened in the middle school grades—the time period linking students' arithmetic and early algebraic reasoning and their development of increasingly complex, abstract algebraic reasoning. In this paper we presented results concerning students' understanding of two fundamental algebraic ideas—equivalence and variable—and the relationship of their understanding to performance on problems that require use of these two ideas. It is our hope that these results will inform the work of both teachers and curriculum developers, so that they can each provide more opportunities for students to develop their understanding of these core concepts.

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