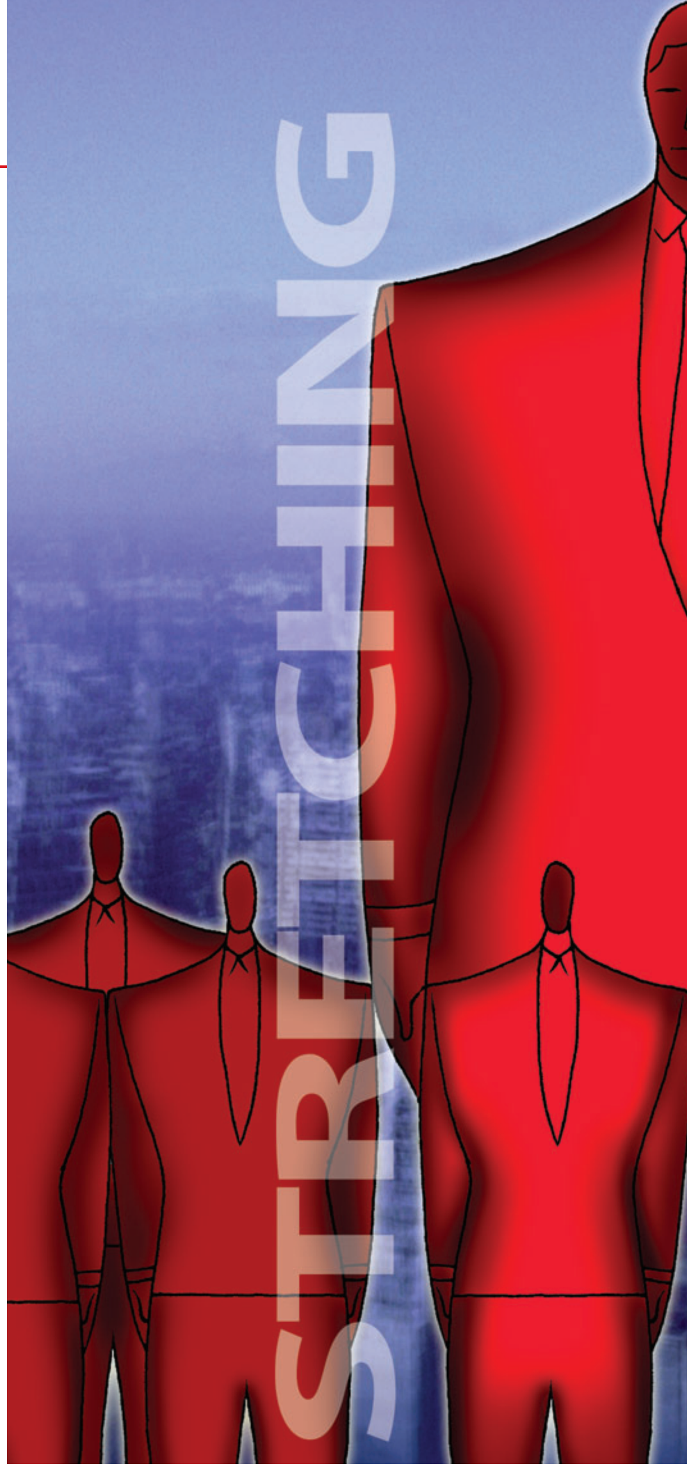
LAURA GRANDAU AND ANA C. STEPHENS

# Algebraic Thinking and Geometry




RESEARCH ON THE LEARNING AND TEACHING of algebra has recently been identified as a priority by members of the mathematics education research community (e.g., Ball 2003; Carpenter and Levi 2000; Kaput 1998; Olive, Izsak, and Blanton 2002). Rather than view algebra as an isolated course of study to be completed in the eighth or ninth grade, these researchers advocate the reconceptualization of algebra as a strand that weaves throughout other areas of mathematics in the K–12 curriculum.

Most work on this "reconceptualization" has occurred in the early elementary grades. Researchers have identified core algebraic ideas that students can begin to engage in as early as the first grade. These core ideas include equality (Carpenter, Franke, and Levi 2003), operation sense (Schifter 1999), and generalization (Blanton and Kaput 2003). Although the elementary grades offer a natural starting place for this work, we hope in this article to contribute to the growing body of work being done at the middle school level about developing teachers' and students' algebraic thinking.

We, the authors of this article, and colleagues from the Supporting the Transition from Arithmetic to Algebraic Reasoning Project at the University of Wisconsin—Madison, have been involved with professional development activities that, to borrow Blanton and Kaput's (2003) phrase, are focused on developing teachers' "algebra eyes and ears." As such, we are working to help teachers recognize the potential offered by tasks to engage students in algebraic thinking, recognize algebraic thinking demonstrated by students, and

LAURA GRANDAU, *ljgrandau@wisc.edu, is a doctoral student in the Department of Curriculum and Instruction at the University of Wisconsin—Madison. Her major areas of research are teacher education and mathematics. She is also interested in international approaches to teaching mathematics, practitioner research, and issues related to linguistic diversity in the classroom.* ANA STEPHENS, *acstephens@wisc.edu, is a researcher at the University of Wisconsin—Madison. She is interested in the development of students' and teachers' algebraic reasoning.*


elicit such thinking through question posing and task extension.

One challenge associated with working toward such a goal at the middle school level is the frequent district- or schoolwide expectation that teachers closely follow a particular curriculum. In the case of our teachers, that curriculum is the Connected Mathematics Project (CMP) (Lappan, Fey, Fitzgerald, Friel, and Phillips 2002). Although we are aware that such an expectation frequently exists at the elementary level as well, this feature of teachers' working lives is rarely if ever alluded to in the early-algebra literature. Given this challenge, we chose to design an activity that would encourage the development of teachers' algebra eyes and ears within the context of their curricular materials. This article describes the work experiences of two teachers.

## The Algebra Project

GROUPS OF TEACHERS AT THE SAME GRADE level worked together to first identify a CMP lesson that offered potential for algebraic thinking—particularly one that was not explicitly part of the program's "algebra" strand. Algebraic thinking could occur in lessons discussing the meaning of the equal sign, the use of different representations, and where students could be asked to symbolize generalizations initially expressed in words.

Teachers next developed probes, questions, or investigations that they believed would elicit algebraic thinking, carefully integrated them into the lesson, and taught the new lessons in their individual classrooms. After implementation, they then reflected on, reacted to, and led a discussion about the lessons, paying particular attention to the integrated algebraic components. Finally, lessons were revised and shared with all teachers in the professional development group.

## Algebra and Geometry

TWO SEVENTH-GRADE TEACHERS, WORKING IN different groups, chose to add algebraic concepts to lessons from a geometry unit in the CMP curriculum called "Stretching and Shrinking." The investigations in this unit are designed to help students acquire the knowledge and experience necessary to reason and make important distinctions about scale and similarity in geometry situations. Lessons focus on learning to identify corresponding parts of similar figures, describing and producing transformations, analyzing and applying scale factors of figures, and applying properties of similar figures. Measurement skills, proportional thinking, and experience working with equivalent ratios are also developed in this unit.

The first teacher, Suzanne, chose a lesson from the investigation called "Patterns of Similar Figures." She saw opportunities to *embed* algebraic concepts and discussion within the structure of the CMP lesson as it was written, concepts that she believed would further support the goals of the lesson. The second teacher, Kelli, saw opportunities to elicit algebraic thinking by *extending* the CMP investigation "Using Similarity." She believed her extended activities could lead students toward understanding ideas beyond the immediate goals of the lesson. Both teachers were enthusiastic about *integrating algebra into* their chosen lessons, implementing them, and sharing reactions with the rest of the professional development group. How the lessons progressed and what they learned are described below.

### Suzanne's lesson

Suzanne chose a lesson introducing students to the concept of a "rep-tile"—defined as a shape whose copies can be reconfigured to make a larger, similar shape (see **fig. 1**, for example). Students are given a set of shapes and are asked to find four identical polygons that can be put together in such a way that the resulting larger polygon is similar to the original. They are then to sketch this larger figure, showing how the pieces fit together, and record the scale factor—the ratio of side lengths of the new shape to the side lengths of the original shape. This process is repeated for each different shape in the set and then repeated again with each of the resulting shapes. For example, a shape consisting of four triangular tiles is now the new rep-tile, and the larger shape consists of sixteen triangles (see **fig. 2**). The overarching goal of the activity is to help students see the relationship between the scale factor and the number of copies of an original polygon needed to make a larger, similar polygon.

Suzanne identified as the most important mathematical features of the lesson (a) the understanding that the area of the new polygon can be obtained by multiplying the area of the original polygon by the square of the scale factor and (b) the recognition of

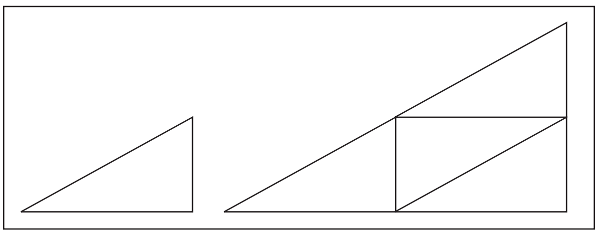


**Fig. 1 A rep-tile and a larger, similar figure consisting of four copies**

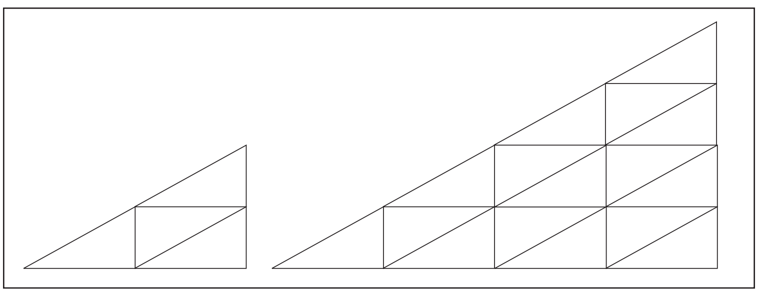


**Fig. 2 A larger rep-tile and an even larger similar figure consisting of four copies**

| | # of Rep-Tiles Used | Scale Factor Small to Large | Scale Factor Large to Small | Area |
|---|---|---|---|---|
| Shape 1 | 1 | 1 | 1 | 1 |
| Shape 2 | 4 | 2 | ½ | 4 |
| Shape 3 | 9 | 3 | ⅓ | 9 |
| Shape 4 | 16 | 4 | ¼ | 16 |
| Shape *X* X | $X^2$ | X | $1 \div X$ | $X^2$ |

**Fig. 3 A student's table of data, including expressions for the general case**

patterns across the collected data. To encourage students to systematically organize their data and find patterns, she asked them to make a table documenting their constructed figures, the number of rep-tiles used, the scale factor that would take them from the rep-tile to the new figure, the scale factor that would take them from the new figure to the rep-tile, and the area of the new figure (see **fig. 3**). Suzanne added algebraic thinking to the lesson by asking students to consider the general case. That is, students were asked to complete the table for a figure with a side length of $x$ tiles. These activities were not part of the original lesson. Suzanne included these tasks because she believed the structure and use of the table and the general case would contribute to her students' abilities to reason algebraically while furthering their understanding of the relationship between scale factor and area. The modifications she made were easily embedded within the existing lesson plan.

After students explored their rep-tiles, constructed similar figures with four identical rep-tiles, used these new figures to construct even larger similar figures, and completed their tables, they were asked to share the patterns they observed. Two students began the discussion by drawing the first and second figures in their tables and reporting the small-to-large and large-to-small scale factors as well as the areas of each figure. Another student moved the discussion to the general case by explaining how he filled in "row $x$" of his table (see **fig. 3**) by looking at the patterns across each row. He noted, for example, that the number of rep-tiles used and the area are always the same and that both are the square of the shape number (or side length). He also stated that the small-to-large scale factor is the same as the side length and that the large-to-small scale factor is always 1 over the small-to-large scale factor (i.e., the reciprocal).

Suzanne reported being pleased that students could create tables and recognize the patterns found within. She was also pleased that some students were able to generate the general case (i.e., row $x$). When asked what she would do differently in the future, Suzanne stated that she would not have students start with only one rep-tile, because doing so resulted in all 1s across the students' first row of data (see **fig. 3**). "Students did not understand that the scale factor needed to be squared and because the area of the original was 1, they did not [initially] think the area of the original mattered." She went on to mention that she would like to spend more time earlier in the school year helping students recognize visual patterns as a way to better prepare them for this lesson. Overall, she was pleased with the implementation of the lesson and the *algebraic* features she embedded to support its goals.

### Kelli's lesson

Kelli developed her algebra and geometry lesson by extending an existing CMP problem about rectangles and similarity. In the original CMP problem (shown in **fig. 4**), students are asked to sort given rectangles into sets of similar rectangles. They are expected to use side lengths, marked in unit squares, to make comparisons. Kelli created an introductory activity (see **fig. 5**) and asked her students to complete that first. She next extended the original problem to include work with graphs, lines, and equations. Her new, integrated lesson was implemented after previous lessons when students had investigated and discussed similarity, scale factor, and the relationship between scale factor and area.

Working independently or with a partner, students started the lesson by plotting points on a coordinate grid and connecting them to make four different rectangles **(fig. 5)**. Students then identified and explained which rectangles were similar and which were not. Next, as a class, they discussed conditions for similarity. In this discussion, students reported that similar shapes (a) could be built by multiplying the side lengths by the same number (enlarging or reducing by the same scale factor), (b) had the same general shape, and (c) had corresponding angles that are equal.

Kelli asked her students to list the coordinates of the top-right vertices of the similar rectangles in the table on the handout. She then asked, "Is there anything interesting about these points?" One student responded, "You can connect them to make a diagonal line." Another said, "The line can help us know if they're similar." Kelli said, "Yes, this line is useful. What else can the line help us know?" A different student said, "It can help us make other similar rectangles." Kelli said that these answers were all correct and drew new similar rectangles on the graph, using points on the line as her guide, saying "We could use this line to make an infinite number of similar rectangles."

She asked the students, "What if we wanted to write a rule or equation that describes that line? Any ideas?" The class discussed the multiplicative pattern they saw in the table, and one student volunteered, "You just square $x$. No, I mean double $x$, like $x \times 2$." Kelli went to the table and led the class through checking the $x \times 2$ statement. In the end, students agreed that $y = 2x$ accurately described the line.

In the second half of the lesson, Kelli asked students to form small groups and look at the original CMP problem in their books **(fig. 4)**. She asked each group to measure and cut construction paper models of the rectangles and to sort them into sets of similar shapes. She told them they were going to work with this information as they did in the activity just completed but that instead of her leading the follow-up discussion, each group would prepare a

Sort the rectangles below into sets of similar rectangles. Describe the method you use.

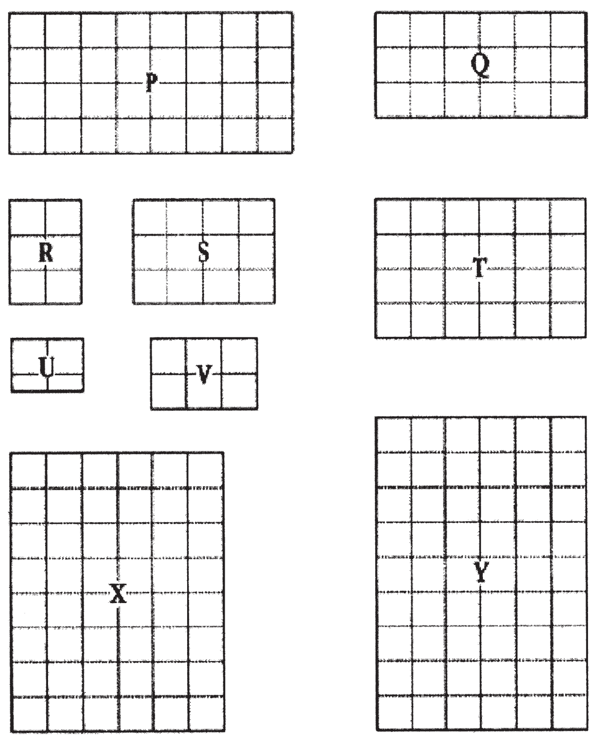


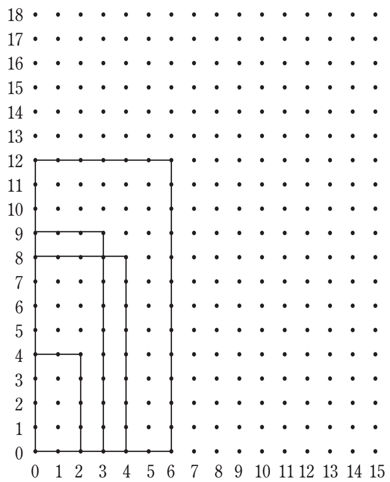



**Fig. 4 A CMP similar figures task**

## Similar Rectangles Warm-up

Draw 4 rectangles on the grid below. (Plot the 4 points, then connect the points using a different color for each set).

A. (0, 0) (6, 0) (6, 12) (0, 12)
B. (0, 0) (2, 0) (2, 4) (0, 4)
C. (0, 0) (3, 0) (3, 9) (0, 9)
D. (0, 0) (4, 0) (4, 8) (0, 8)



Which rectangles are similar?

______________________

Which rectangles are not similar?

______________________

How do you know?

______________________

List the coordinates of the top right vertex of each rectangle.

| $x$ | $y$ |
|---|---|
| | |

Fig. 5 Kelli's warm-up activity

presentation and explain what was discovered about its set of similar rectangles.

During this part of the lesson, students traced one of their new sets of similar rectangles onto large grid paper. As in the first activity, they then drew a line through the upper-right vertices of these rectangles; most students situated the lower-left vertices at (0, 0). Then, with the help of the graph, they made tables of the $x$- and $y$-coordinates of the upper-right vertices and looked for patterns. Finally, students were asked to write an equation for the line on their grids. Most student groups were able to find a pattern by looking at the points in their table, but not all were successful writing an equation. As a whole class, students used the graphs and lines to predict the dimensions of other similar rectangles. With this extended activity, Kelli hoped that students might see how using algebraic thinking could help them organize information in meaningful ways and make good predictions.

Kelli identified the most important mathematical components of the overall lesson to be (a) the understanding of geometric similarity through practice with shapes, measure, scales, and coordinate graphs and (b) the ability to create a table using data about similar rectangles and write an equation that fits the data. Although not a mastery goal of the lesson, Kelli also wanted to see if students could make accurate predictions about other similar rectangles using the information collected to solve the original problem. The algebraic concepts she hoped her students would engage with included the use of patterns and graphs, the generation of a table, work with equations, and a beginning sense of how algebra can be used as a predictive tool.

After the lesson was implemented, Kelli reported being pleased. She said the first activity was useful, and students began thinking about the features of geometric objects that make them either similar or not. She believed students found the line connecting the upper-right vertices useful and that they understood what it represented in terms of similar rectangles. Regarding the second half of the lesson, Kelli said the construction paper models she added to the problem were also helpful in offering students the chance to manipulate and move the shapes when making comparisons. She also believed that in this part of the lesson, many students could see more clearly how the line was formed on the graph and how it was related to patterns and predictions.

Kelli believed, however, that during the algebra and geometry lesson, "not *all* students were clear on the line—I thought some might be thinking, 'Why am I doing this?' " In the future, Kelli said, she would need to more clearly explain to students how a line can be used to make predictions before she asks them to draw it. She also said that perhaps more work with equations in general would be useful, as most groups needed help writing an accurate equation. Finally, Kelli noted that by extending this CMP problem to include certain algebraic ideas, she made the lesson considerably longer and more complex. She knew this would make it more challenging for herself and her students and said that in a future implementation, she would devote more time to the task and ask more in-depth questions to help all students make connections. Overall, she was pleased with her experience with the extended *algebraic* features of the lesson and believed her students benefited from working with them as well.

## Discussion

SUZANNE'S AND KELLI'S STORIES REINFORCED for us the idea that algebra can in fact be integrated with other content areas—in this case, geometry—

within the context of teachers' existing curricular materials. In these two cases, students were thinking algebraically when they were using graphs to make predictions about similar figures, writing linear equations to describe these graphs, making verbal generalizations about features of geometric figures, and expressing these generalizations in symbols. The teachers themselves were exercising their own mathematical knowledge and building connections between algebra and geometry. These teachers furthermore provided illustrations of two different models that one could follow when integrating algebra with other content areas. In Suzanne's case, algebraic reasoning was *embedded* in the existing lesson such that it contributed to the goals of the lesson as originally written. In Kelli's case, on the other hand, algebraic reasoning *extended* students' thinking beyond the original goals of the lesson.

The fact that both teachers identified changes they would make to their lessons to improve future implementations highlights the importance of thoughtful planning and follow-up reflection. They both spent much time determining where and how algebraic ideas might fit into their lessons. In the end, Suzanne and Kelli agreed that integrating algebraic concepts into existing tasks takes strong mathematical knowledge as well as time to help students develop prerequisite skills, complete in-depth questioning, and assess students' learning.

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*The research reported is supported in part through a grant from the Interagency Educational Research Initiative (IERI), No. REC-0115661. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the funding agencies.*

*The authors wish to thank Pamela Asquith for her contributions to this project, Suzanne Folberg and Kelli Tuttle for welcoming them into their classrooms, and Tom Blau for his contributions to Kelli's lesson.* □