Investigating middle-school teachers' perceptions of algebraic thinking

Pamela Asquith
Ana C. Stephens
Laura Grandau
Eric J. Knuth
Martha W. Alibali

University of Wisconsin-Madison

American Educational Research Association
Montreal, Canada
April 2005

Abstract

This study explored ten middle-school teachers' perceptions of algebraic thinking by asking them to identify opportunities offered by particular tasks to engage students in algebraic thinking, predict student responses, and describe algebraic features of students' written work. Results indicate these teachers had language to describe algebraic concepts, identified different levels of algebraic thinking, and described different pedagogies to be used in response. In relation to prior research, we saw that teachers accurately predicted students' most commonly-used approaches. We also observed, however, that several teachers held traditional views of algebraic thinking dominated by symbol use. We believe our findings shed light on the current state of teachers' perceptions of algebraic thinking and may have important implications for teachers' practices and development.

Objectives and rationale

Many mathematics education researchers (e.g., Carpenter, Franke, & Levi, 2003; Kaput, 1998; Schifter, 1999) argue that the study of algebra should begin much earlier than eighth or ninth grade. They believe increased access to algebraic ideas throughout the K-8 curriculum will smooth students' transition from arithmetic to formal algebra by providing an early and necessary understanding of such concepts as equality, operation sense, and generalization. To this end, they call for reform in mathematics teaching and learning that requires elementary- and middle-school teachers to recognize opportunities in their everyday teaching to engage students in algebraic thinking. They promote the development of teachers' pedagogical content knowledge in this area and encourage the analysis of mathematical tasks that might elicit algebraic reasoning.

The goal of our research project *Supporting the Transition from Arithmetic to Algebraic Reasoning [STAAR]* is to investigate middle school students' thinking about core algebraic concepts: equivalence, variable and representational fluency. In addition to gaining a deeper understanding of students' algebraic thinking, we also strive to address the fact that attention to algebraic thinking is new to many middle-school teachers. Accordingly, the focus of our professional development efforts has been to help teachers recognize students' algebraic thinking and opportunities to foster such thinking. These goals were addressed by a two-pronged approach. First, professional development instructors shared what has been learned from *STAAR's* assessments about middle-school students' conceptions and misconceptions about equality (e.g., the equal sign means "the answer" or "the total") and variable (e.g., a variable can stand for a number, but not an expression). Second, they asked teachers to engage in cycles of problem solving, sharing strategies with colleagues giving the same tasks to their students and return to the next class with a collection of student work to examine. Follow-up discussions addressed solution strategies used, difficulties encountered by students and what student responses revealed about student thinking. Designed to inform future professional development efforts, this study investigates whether and to what extent, middle-school teachers attend to opportunities for algebraic reasoning in tasks and recognize it in student work.

Theoretical framework

Our professional development work with middle-school teachers and investigation of their perceptions of algebraic thinking are guided by Blanton and Kaput's (2003) notion of "algebra eyes and ears." Teachers with developed algebra eyes and ears are able to recognize potential offered by tasks to engage students in algebraic thinking; are able to recognize algebraic thinking demonstrated by students; and are able to elicit such thinking through question-posing and task extension. Such a framework addresses the reality that—because algebra is now to be viewed as a "web of knowledge and skill" (Kaput, 1998), rather than as an isolated course of study—elementary- and middle-school teachers must be able to recognize and capitalize on opportunities to engage their students in algebraic thinking as these opportunities arise in the course of everyday teaching.

The specific research questions addressed in this study were the following:

1. What opportunities for engaging students in algebraic thinking do middle-school teachers recognize in particular tasks?

2. What is the nature of middle-school teachers' predictions of student responses to particular tasks? How are these predictions related to students' actual performance?

3. What aspects of written student work do middle-school teachers identify as algebraic?

Method

Participants

The participants in our study were ten middle-school teachers (grades 7–8) from two socio-economically diverse middle schools in a small urban district. These ten included six math teachers and four special education teachers who support students in math class. Teaching experience ranged from 2 to 30 years. These teachers were participants in our professional development course designed to further their abilities to understand and cultivate their students' algebraic thinking. They volunteered early in the school year to take part in the study.

Data Collection

Each teacher participated in an hour-long videotaped interview. This interview was organized around three tasks selected from Supporting the Transition from Arithmetic to Algebraic Thinking [STAAR] assessments of students' algebraic thinking. For each of the 3 tasks, teachers were asked to discuss what potential, if any, the task might have to engage students in algebraic thinking; to propose strategies students might use and discuss difficulties students might have; and to examine samples of student work on the task and describe features of algebraic thinking observed. The focus of this paper will be on results generated by one of these tasks—a generalization task commonly known as the Toothpick Problem (see Figure 1). Several semesters prior to conducting these interviews, 234 seventh- and eighth-grade students at one of the schools represented by our professional development teachers were given the Toothpick Problem as part of a written assessment designed to investigate the development of algebraic thinking across the middle grades. The data generated from students' written work on this task will be discussed as it relates to teachers' predictions of student responses and difficulties.

Results and Discussion

Recognizing opportunities for algebraic thinking

Each teacher identified at least one opportunity provided by the *Toothpick Problem* to engage students in algebraic thinking. In total, the participants' mentioned 19 opportunities for algebraic thinking that were collapsed into five categories as summarized in Table 1.

Participants most frequently noted that the *Toothpick Problem* offered the opportunity for students to write an equation or to recognize patterns. This result is not surprising given the task's demands. Somewhat surprising, however, is that only three participants made any specific reference to the opportunity for generalization, a primary algebraic component of this task. On the other hand, we were encouraged that three participants noted the opportunity the task offered for building representational fluency, as the ability to link and translate among multiple mathematical representations is considered a critical feature in the development of algebraic thinking (Driscoll, 1999).

In addition to discussing features of the *Toothpick Problem* identified as algebraic, some teachers' responses to this question reflected a belief that the opportunities a task like this provides vary according to what a teacher does in the classroom. For example, one seventh-grade teacher explained, "I think you'd end up with some things that could be simplified to $3x + 1$ but wouldn't initially be $3x + 1$ and then I think you could do some talking about how and why these are all the same." This teacher thus recognizes the role she might play in scaffolding her students' understanding of equivalent algebraic expressions.

Predicting student responses

Teachers' predictions of student responses to the *Toothpick Problem* are summarized in Table 2. In response to part b (*How many toothpicks are required to make 7 squares in a row?*), all ten teachers said their students would draw and count because, as one representative teacher stated, "the number [of squares] here is reasonable." Only three of the teachers mentioned an additional strategy. In response to part c (*Write an equation that you could use to figure out how many toothpicks are needed to build any number of squares in a row.*), teachers said students would use a wide range of strategies. Teachers mentioned, for example, that some students "would see the four to start and add three more for each additional square," "say it's three times *n* plus the extra one at the end," make a table or generate more cases. Most teachers said that a number of their students would not be able to accurately write an equation to represent the general case.

It is interesting to note that when describing expected student responses for part c, nine of the ten teachers conceptualized different levels of thinking occurring in the same classroom. In these cases, teachers stated that "some students" would solve the problem in one way and "others" would solve it in another. Teachers made distinctions between "a strong algebra student" or "the more sophisticated kids," and "weaker students" or "others." Coupled with this, teachers described pedagogical moves they would make to advance different students' thinking. One teacher said, for example, "Some students may need to manipulate real toothpicks to make sense of the problem. So I would give them toothpicks. Some of the others don't think in [terms of] equals, they think in terms of answers to the problem, so setting up the equation would require some sort of prompting, like 'what equals what?'"

When asked what difficulties students might have on part b, teachers mentioned few. They believed most students would be successful with this part. Regarding part c, only 1 teacher explicitly stated, "Students would not be able to generalize it." Teachers did however mention three difficulties: forgetting about the shared side; not knowing what to do with the shared side;

and all of the teachers interviewed responded that 7^{th} and 8^{th} grade students would be challenged by the task of writing an equation using symbols. A few teachers also noted that students would find it easier to generalize with words than to write an equation.

In relation to actual student performance on part b, we found that teachers' predictions of student success were accurate. 66% of the 7^{th} graders and 73% of the 8^{th} graders solved the problem correctly. Teachers also accurately predicted the most-commonly used strategy: draw and count. Seventy percent of the students used this strategy. With respect to part c, approximately half of the 234 7^{t-h} and 8^{th} graders who completed the assessment were asked the same question presented to the teachers (*Write an equation that you could use to figure out how many toothpicks are needed to build any number of squares in a row)*. Teachers accurately predicted students would have difficulty with this task. Our data show 41% of students completed it correctly. Teachers' expectation that students would be more successful using words to describe the rule was also supported by our student data as the other half of the 7^{th} and 8^{th} grade students were asked to "*Describe in words how you could figure out how many toothpicks are needed to build any number of squares in a row*." and did so successfully 58% of the time.

Recognizing algebraic thinking in student work

After discussing opportunities the *Toothpick Problem* offers to engage students in algebraic thinking and predicting how students would respond to the task, teachers were asked to examine three pieces of student work on the task (see Figure 2) and classify the work as algebraic or not algebraic and explain their choice. Reminiscent of their discussion of the opportunities the task offers to engage students in algebraic thinking, teachers again focused a great deal on students' use of variables and equation writing. In general, teachers were apt to label work in which students developed explicit formulas (whether exclusively in symbols as shown in Nicole's work or in a word-symbol combination as shown in Eric's work) as algebraic, and the sample of work in which a student described the pattern recursively in words (Aaron's work) as somewhat less algebraic (see Table 3). When describing the written work, teachers tended to cite students' abilities to recognize the underlying pattern and/or state a generalization of this pattern that could apply to any number of toothpicks as indications of algebraic thinking (n = 6 for Eric and Nicole's work, n = 5 for Aaron's work). This is in contrast to teachers' discussion of the task's opportunities, when very few of them explicitly mentioned opportunities to generalize. Another prevalent response, occurring only in reaction to Nicole's work, was that the student was thinking algebraically because she was able to write an equation using a variable (n = 6). One teacher stated, for example, that Nicole has "more of an understanding of algebraic thinking…than the other two." When asked why, she explained:

> Because of the *x*'s, the symbols…. I guess I always think when it's in a formula and it's got your symbols in there, that you're at least a little bit higher in your thought process of how to solve it in terms of algebra, algebraic thinking.

In comparing Eric and Nicole's work, this teacher focused not on the thinking that led to the two different—although correct—generalizations but rather on the use or non-use of algebraic letters. She said Nicole's work was more sophisticated because "it ends up with a formula." Responses such as this are reminiscent of Kaput's (1998) concern that traditional school algebra has focused on symbol manipulation at the expense of all other forms of algebraic reasoning. Another response of interest was that Eric was thinking algebraically because he was "thinking about variables" even though he did not actually use algebraic letters (n = 2).

Recognizing mathematical connections

When asked "*Do you see any mathematical connections among the responses of these three students?*", 4 of the 10 teachers interviewed stated that the expression consisting of a symbol-word combination (# of squares*3 + 1 = # of toothpicks) and that involving formal algebraic symbols ($4x - x + 1 =$ toothpicks) were equivalent. Two of these 4 teachers additionally shared that they would use these two responses as an instructional opportunity asking students questions such as the following:

Are these two equations equivalent?

Is the phrase " # of squares" the same as " x"?

Do these two responses represent the same pattern?

As indicated in student responses to part c students may use a range of notations to represent their generalizations including verbal descriptions, word-symbol combinations, and the use of formal algebraic symbols. Key to designing instruction that supports students' transition from recursive descriptions of patterns to functional thinking is teacher recognition of the connections between these representations (Driscoll, 1999).

Conclusion

Our findings suggest that middle-school teachers hold multiple perceptions of what it means for a student to think algebraically. On the one hand, teachers identified different aspects of algebraic thinking in students' written work, and used terms such as "generalization" and "representation" in describing it. They were able to fairly accurately predict what students might do in response to a particular task. They also offered a range of possible strategies different levels of students might use and a variety of possible pedagogical moves they as teachers might make in response. On the other hand, there were indications that several teachers hold the perception that strong algebraic thinking involves the use of variables. As one teacher said of algebraic thinking, “Part of it is always going to be that they use the symbols, they use letters.” Implications of these findings include the need to broaden teachers’ conceptions of what it means for students to think algebraically so that their focus shifts away from particular representations (e.g., symbol use is inherently algebraic) and towards the student thinking behind these representations. Teachers who understand these links will be better equipped to facilitate student connections between representations.

References

Blanton, M. L., & Kaput, J. J. (2003). Developing elementary teachers' "algebra eyes and ears". *Teaching Children Mathematics, 10*(2), 70-77.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grades 6-10*. Portsmouth, NH: Heinemann.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In S. Fennel (Ed.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a National Symposium* (pp. 25-26). Washington, DC: National Research Council, National Academy Press.

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking in grades K-6. In L. V. Stiff (Ed.), *Developing mathematical reasoning in grades K-12* (pp. 62-81). Reston, VA: National Council of Teachers of Mathematics.

Table 1
Opportunities identified in the Toothpick Problem for engaging students in algebraic thinking.

Opportunity identified	Number of Participants	Example response*
Equations/expressions	7	To see the pattern, extend the pattern to the next step, and then of course at the end to make the equation to extend it to any number of squares. (7th grade math teacher)
Patterns	5	They've got to recognize patterns and…looking for different ways to represent those patterns involves using expressions and so I think that would be the connection between this and moving into algebraic thinking. (8th grade math teacher)
Generalization	3	I think it gives them an opportunity to generalize one problem and generalize it to the point where they can figure out any number of squares given the information they have on two or three squares. (7th grade special ed teacher) I suppose you could just expand on this and you could expand on the number of boxes and you could probably take different shapes…like triangles. (7th grade math teacher)
Representational fluency	3	If [students]…write an equation and they know they have strategies to get an equation by making a table…then they're going back and forth between the different representations….Or, if they graph their table, they can figure out an equation that way. (8th grade special ed teacher)
Variables/symbols	1	Well this is a good one because…it really gives [students] a motivation to learn about using symbols…so this seems like a good introduction to the symbolic form. (8th grade math teacher)

*Categories are not mutually exclusive (e.g., the example coded as "equation/expression" above was also coded as "generalization"). Each excerpt was selected from a different participant's transcript.

Table 2
Teachers' expectations of students' responses to the Toothpick Problem.

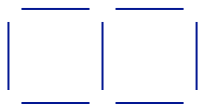
Expected responses	Number of Participants
Part b	
How many toothpicks required to make 7 squares in a row?	
Draw and count	10
Make a table/find a pattern	3
Part c	
Write an equation that you could use to figure out how many toothpicks are needed to build any number of squares in a row.	
Generalize in words	3
Write an equation with symbols	2
Make a table/find a pattern	2
Say "I don't know" or "be stuck"	2
Generate more cases	1
Be unable to generalize	1
Count all and subtract overlapping sides	1

Table 3

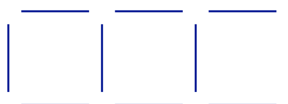
Participant responses to "Would you say this student was thinking algebraically?"

	Eric	Aaron	Nicole
Yes	10	5	9
No	0	2	1
Starting to	0	3	0

If a side of each square in the shape below is 1 toothpick, then it takes 7 toothpicks to make 2 squares in a row.



a) How many toothpicks are required to make 3 squares in a row?

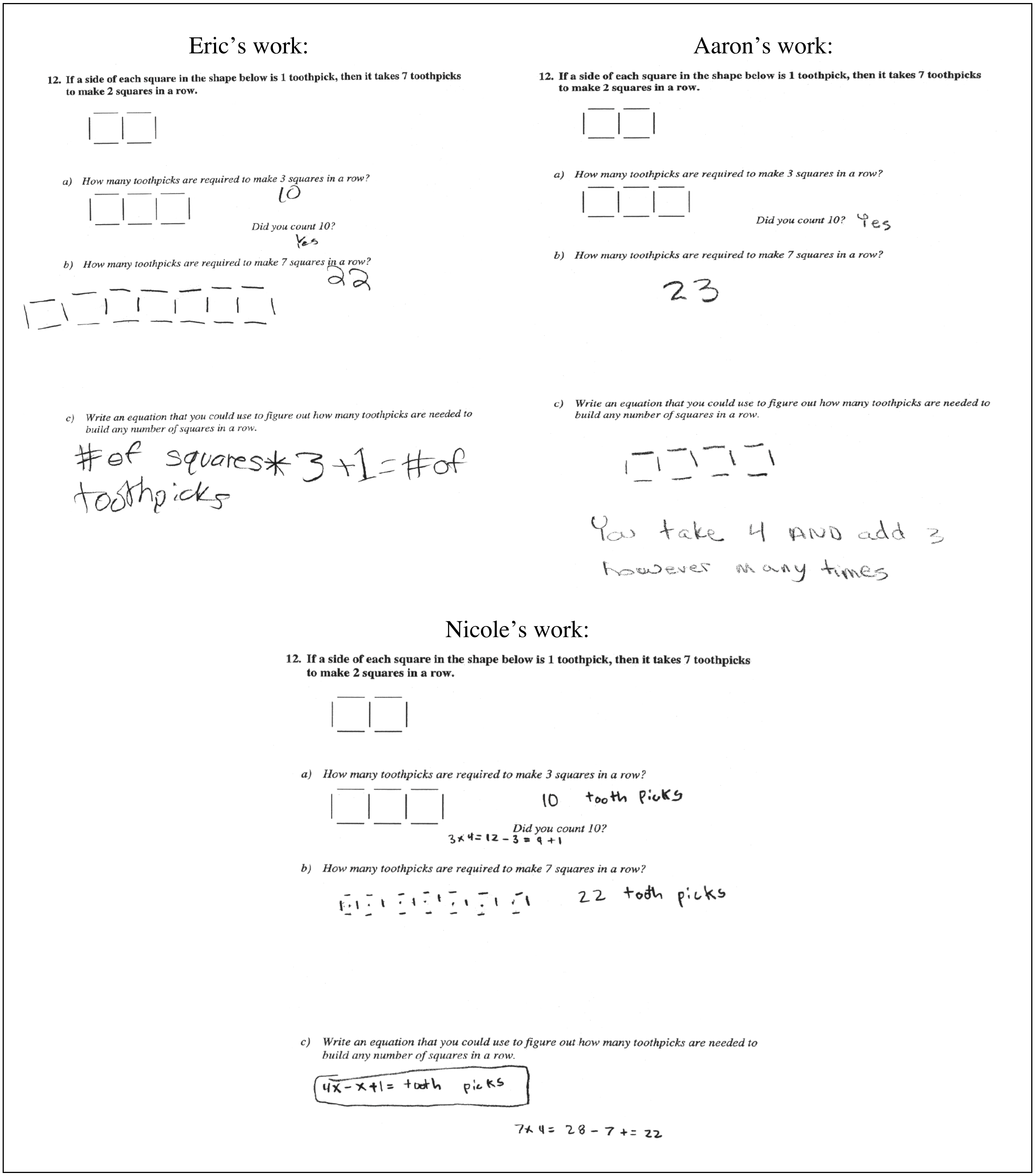


Did you count 10?

b) How many toothpicks are required to make 7 squares in a row?

c) Write an equation that you could use to figure out how many toothpicks are needed to build any number of squares in a row.

Figure 1. The *Toothpick Problem*.



Eric's work:

12. If a side of each square in the shape below is 1 toothpick, then it takes 7 toothpicks to make 2 squares in a row.

a) How many toothpicks are required to make 3 squares in a row?

10

Did you count 10?

Yes

b) How many toothpicks are required to make 7 squares in a row?

22

c) Write an equation that you could use to figure out how many toothpicks are needed to build any number of squares in a row.

#of squares*3+1=#of toothpicks

Aaron's work:

12. If a side of each square in the shape below is 1 toothpick, then it takes 7 toothpicks to make 2 squares in a row.

a) How many toothpicks are required to make 3 squares in a row?

Did you count 10? Yes

b) How many toothpicks are required to make 7 squares in a row?

23

c) Write an equation that you could use to figure out how many toothpicks are needed to build any number of squares in a row.

You take 4 AND add 3 however many times

Nicole's work:

12. If a side of each square in the shape below is 1 toothpick, then it takes 7 toothpicks to make 2 squares in a row.

a) How many toothpicks are required to make 3 squares in a row?

10 tooth picks

Did you count 10?

3x4=12-3=9+1

b) How many toothpicks are required to make 7 squares in a row?

22 toth picks

c) Write an equation that you could use to figure out how many toothpicks are needed to build any number of squares in a row.

4x-x+1= tooth picks

7x4= 28-7+= 22

Figure 2. Three samples of student work on the *Toothpick Problem*.